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(10) JANET MYHRE

PRINCIPAL INVESTIGATOR

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TABLE OF CONTENTS

Final Report for ONR Grant N00014-76-C-0695

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Janet Myhre
Principal Investigator

During the period of this grant significant progress has been made in determining probabilistic and statistical properties for the mixed exponential distribution (Pareto Type II distribution). This probability law is proving to be an accurate model for fitting the distribution of life lengths for most types of electronic equipment. The research related to the mixed exponential has resulted in:

1. "Comparison of Parameter Estimation for the Pareto Distribution," submitted to the Journal of the American Statistical Association. This paper shows that the maximum likelihood estimate for the scale parameter of the mixed exponential distribution, using complete or censored data, is at least as accurate as BLUE (best linear unbiased estimate) or modified BLUE. In addition, it is shown that the maximum likelihood estimates are more versatile than the BLUE and modified BLUE. [Attachment C].

2. "Asymptotic Distribution of Maximum Likelihood Estimates for Parameters of the Mixed Exponential Distribution based on Censored Data." Draft in preparation for submission to Technometrics for publication together with:

3. "Problems of Estimation for a Decreasing Failure Rate Distribution Applied to Reliability," which has been submitted to Technometrics and is now under revision. In this paper the properties of mixed exponential distribution are studied. Simple

techniques are derived which yield sufficient conditions for obtaining the solution of the maximum likelihood equations when one or both of the parameters are unknown, even when the data is highly truncated or censored. [Attachment B].

4. "Approximate Confidence Bounds for Reliability; Mixed Exponential Distribution." A draft of this paper, minus computations of the computer results, has been prepared. Additional simulations will be run in order to complete the study. The simulations run to date show that in general the bounds are quite accurate and that the variance of the distribution of bounds is relatively small. This paper will be submitted to Technometrics or Naval Logistics Quarterly. [Attachment F].

5. "Robustness of the Mixed Exponential Distribution in Fitting Mixtures of Exponentials." Computer programs have been written which, for known mixtures of exponential distributions, check the comparative accuracy of fit among the Mixed Exponential distribution, the Weibull distribution and the Exponential distribution. Work to date shows that under mixtures quite different than the assumed Gamma mixture, the Mixed Exponential is still fairly robust. This paper will be submitted to the Journal of Statistics and Simulation.

6. "HP-97 Programs for the Computation of the Maximum Likelihood Estimates for the Parameters of the Mixed Exponential Distribution." This program makes the Mixed Exponential model more available to users. It will allow the estimation of parameters for small to moderate sample sizes. [Attachment D].

Additional research completed under the grant includes: "Determining Confidence Bounds for Highly Reliable Coherent Systems based on a Paucity of Failures." This paper has been accepted for publication by the Naval Logistics Quarterly. It deals with a computationally simple method for obtaining confidence bounds for highly reliable coherent systems, based on component tests which experience few or no failures. Binomial and Type I censored exponential failure data are considered The component unreliabilities are ordered by weighting factors which are presumed known. Sensitivity of the confidence bounds to these assumed weights is examined and shown to be low. [Attachment A].

An invited tutorial paper was delivered at the 1977 American Society for Quality Control Technical Conference in Philadelphia. It deals with decreasing failure rates and the Mixed Exponential Distribution. [Attachment E].

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ATTACHMENT A

DETERMINING CONFIDENCE BOUNDS FOR HIGHLY RELIABLE COHERENT SYSTEMS BASED ON A PAUCITY OF COMPONENT FAILURES*

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0. Abstract

A computationally simple method for obtaining confidence bounds for highly reliable coherent systems, based on component tests which experience few or no failures, is given. Binomial and Type I censored exponential failure data are considered. The component unreliabilities are ordered by weighting factors which are presumed known. Sensitivity of the confidence bounds to these assumed weights is examined and shown to be low.

1. Introduction

Previously, confidence bounds for general coherent structures have been obtained by using asymptotic methods, such as Likelihood Ratio [6], Maximum Likelihood [8], or Modified Maximum Likelihood [1], by using Bayesian methods [7], or by assuming equal reliabilities for all components. Asymptotic methods may be inaccurate at higher percentiles unless the number of failures

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*Research, in part, supported by the Office of Naval Research Contract N00014-76-C-0695.

is very large. With Bayesian methods the possibility of inadvertently influencing a decision through the selection of a prior distribution, when the number of failures is small, is well known. Finally, because the assumption of equal reliabilities of the components may not always be fulfilled, the accuracy of a bound obtained using this assumption could be in doubt. What we propose here is to use engineering knowledge which can be gained from accelerated life tests, material qualification tests, or laboratory tests of components. This knowledge can be utilized in a manner that provides information about the parameter space. It is felt that this intermediate ground avoids some of the objections raised by Bayesians concerning "classical" statisticians who operate under an assumption of total ignorance about the parameter space. Moreover, it attempts to avoid the subjectivity which seems to hinder the acceptance of Bayesian methods.

For special structures, such as series structures (and in some cases parallel and series parallel structures) exact methods [9] and additional approximate and asymptotic methods ([4], [5], [9]) for obtaining system confidence bounds have been developed. The accuracy of these approximate bounds has been studied in specific sample cases for structures of order two or three ([5], [9]). In this paper some comparisons are made between the bounds obtained by the weighting method developed here and approximate (asymptotic) bounds for special structures where approximate (asymptotic) bounds can be calculated.

2. Binomial Component Failure Distributions

Since the unreliability of any practical system must be low, that of any component must be even lower. The commonly used technique of obtaining confidence bounds for the probability of success from sequences of Bernoulli trials will not be applicable here, because virtually all of the components will have experienced no failures during their acceptance testing. Extending an idea utilized by Lomnicki [3], we will examine the probability of system failures expressed in terms of the least reliable component. The estimate of this quantity is then used to construct a lower confidence bound on the system reliability.

A qualification test for each component consists of a number of Bernoulli trials of nominally identical components. Given there are n_i trials with X_i failures for the i^{th} component, then it is assumed that the number of failures has a binomial distribution where q_i is the unreliability and n_i is the number of observations. We denote this by

$$X_i \sim B(n_i, q_i) \qquad \text{for } i=1,2,\ldots,m.$$

Assume that q_i may be expressed for each i by

$$(2.1) \qquad q_i = a_i q \quad \text{where } q = \max_{i=1}^{m} q_i \quad \text{and } 0<a_i\leq 1.$$

For the present assume that the a_i are known apriori. In practice we have obtained the a_i from reliability goals and prediction reliabilities. In order to obtain a confidence bound for q, we make the following assumptions. Since the q_i are small the distribution of X_i may be accurately approximated

by a Poisson distribution with mean $\lambda_i = n_i q_i$. Thus, assume

$$X_i \sim P(\lambda_i) \equiv P(n_i q_i). \tag{2.2}$$

(For q_i as large as .01, this approximation is valid for n_i as small as 10.)

An upper confidence bound for q is obtained in the usual way since

$$\sum_{i=1}^{m} X_i \sim P\left(\sum_{i=1}^{m} \lambda_i\right)$$

where

$$\lambda \equiv \sum_{i=1}^{m} \lambda_i = \sum_{i=1}^{m} n_i q_i = q \cdot \sum_{i=1}^{m} a_i n_i .$$

The 100β% upper confidence bound for λ, call it λ_u, is the value of b for which

$$\sum_{j=0}^{k} e^{-b} b^j / j! = 1-\beta \text{ , where } k = \sum_{i=1}^{m} X_i .$$

It follows that the 100β% upper bound for the unreliability q, call it q_u, is

$$q_u = \lambda_u / \sum_{i=1}^{m} a_i n_i . \tag{2.3}$$

We point out that the Poisson approximation to the Binomial as used above is not necessary for the calculation of this type of bound; however, it does greatly facilitate the computation.

In order to obtain a lower confidence bound for the reliability of a coherent system with component reliabilities $\underline{p} = (p_1, \ldots, p_m)$, let

$$p_i = 1-q_i = 1-a_i q \quad \text{for } i=1,\ldots,m$$

where a_i and q are defined as in equation (2.1). System

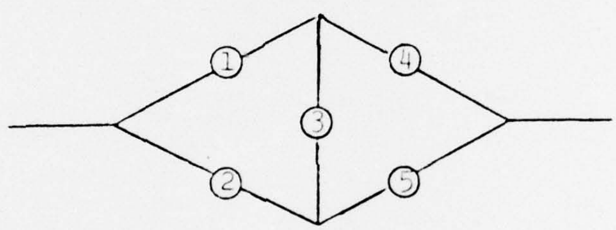
reliability, $h(\underline{p})$, may now be expressed as a function of q alone. Let the induced function be denoted by the equation

$$h(\underline{p}) = h(1-a_1q,\ldots,1-a_mq) = \mathcal{h}(q;\underline{a})$$

where $\underline{a} = (a_1,\ldots,a_m)$. The function $\mathcal{h}(q;\underline{a})$ is strictly decreasing as a function of q. Hence a lower confidence bound on reliability, $\mathcal{h}(q,\underline{a})$, is $\mathcal{h}(q_u;\underline{a})$ where q_u is an upper confidence bound on q.

To illustrate these concepts, consider the following examples.

<u>Example 2.1</u>: For the following bridge structure of five independent components:



if the component reliabilities are p_i for i=1,...,5 then the system reliability is given by

$$\begin{aligned} h(\underline{p}) = {} & p_1p_4 + p_2p_5 + p_1p_3p_5 + p_2p_3p_4 \\ & - p_1p_2p_3p_4 - p_1p_2p_3p_5 - p_1p_2p_4p_5 - p_1p_3p_4p_5 \\ & - p_2p_3p_4p_5 + 2p_1p_2p_3p_4p_5. \end{aligned}$$

Rewriting in terms of the unreliabilities $1-p_i = q_i = a_iq$ for i=1,...,5 yields

$$\begin{aligned} \mathcal{h}(q,\underline{a}) = {} & 1-q^2(a_1a_2+a_4a_5)-q^3(a_1a_3a_5+a_2a_3a_4) \\ & +q^4(a_1a_2a_3a_4+a_1a_2a_3a_5+a_1a_2a_4a_5+a_1a_3a_4a_5+a_2a_3a_4a_5) \\ & -2q^5(a_1a_2a_3a_4a_5). \end{aligned}$$

From engineering analyses it is known that components 1, 2, 4 and 5 have the same unreliability. However, it is

assumed that component 3 is only 3/10 as unreliable as the other components. Assume the following weights and test results.

component	a_i	n_i	x_i
1	1	10	0
2	1	10	0
3	.3	20	0
4	1	10	0
5	1	10	0

Suppose a 90% confidence bound is desired. Since $k = \sum_{i=1}^{5} x_i = 0$, λ_u is the value of b for which $e^{-b} = .10$, so $\lambda_u = 2.303$. Hence an upper bound on the unreliability q, at the 90% level, is given by $q_u = \lambda_u / \sum_{i=1}^{5} a_i n_i = 2.303/46 = 0.050$. Finally, the desired 90% lower confidence bound on reliability is given by $h(q_u, \underline{a}) = 0.995$.

For bridge structures it is not possible to compute Approximately Optimum [4] or Poisson Approximation [9] bounds. The asymptotic methods are generally not applicable unless failures are observed.

Example 2.2: Assume a series structure of order five has known weighting factors a_i and sample sizes n_i where i=1,2,...,5. For industrial problems we have often found that sample sizes are not equal but are roughly proportional to the unreliabilities with the most unreliable component having the smallest sample size. One reason for this may be that specialized, complex equipment often tends to be both unreliable and expensive.

component	a_i	n_i
1	1/2	40
2	1	20
3	1/4	80
4	1/2	40
5	1/2	40

Assuming one failure on component 2, $\lambda_u = 3.89$ at the 90% confidence level. Using (2.3) we find that $q_u = .0389$. Since $h(q;\underline{a})=\prod_{i=1}^{5}(1-a_i q)$, our confidence bound on system reliability is

$$h(q_u;\underline{a}) = .897 \text{ .}$$

The bounds obtained by approximate or asymptotic methods [4], [5] and [9] are much lower than this bound, for example

Approximately Optimum (AO) Bound = .820

Modified Maximum Likelihood Bound (MMLI) = .819

Poisson Approximation (PA) Bound = .806

3. Sensitivity of Confidence Bounds to Assumed Weights

The question that arises is, what is the real difference between the bounds obtained presuming that $\underline{a}$ is known when in fact it may not be. A measure of the error caused by this supposition upon the bound obtained should be found. Let the estimates made by the experimenter for the values of $\underline{a}=(a_1,\ldots,a_m)$ be denoted by $\underline{\alpha}=(\alpha_1,\ldots,\alpha_m)$. The estimate of the upper bound constructed using $\underline{\alpha}$ in equation (2.3) will be denoted by

$$\tilde{q}_u = \lambda_u / \sum_{i=1}^{m} \alpha_i n_i \text{ .} \tag{3.1}$$

Example 3.1:

Differences between the exact bounds obtained in example 2.3 and bounds obtained using various $\underline{\alpha}$ are given below. The corresponding AO, MMLI and PA bounds are also given. (The asterisk by the a_i or α_i indicates the component on which the failure is assumed to have occurred.)

Component	n_i	a_i	$\alpha_i^{(1)}$	$\alpha_i^{(2)}$	$\alpha_i^{(3)}$	$\alpha_i^{(4)}$	$\alpha_i^{(5)}$
1	40	1/2	1	1/2	1	1/5	1/100
2	20	1*	1/2	1/4	1	1*	1*
3	80	1/4	1/4	1*	1*	1/10	1/100
4	40	1/2	1*	1/2	1	1/5	1/100
5	40	1/2	1	1/2	1	1/5	1/100

Bounds						
Weighting Method	.897	.906	.928	.915	.878	.817
AO	.820	.871	.882	.882	.820	.820
MMLI	.819	.916	.950	.950	.819	.819
PA	.806	.806	.806	.806	.806	.806

Note the weighting in case 5 which must be assumed in order to obtain confidence bounds which are close to the AO or MMLI methods. In general, we would not expect the engineering estimates of the α_i to be this different from the correct weights.

We now introduce the following notation for subsequent use:

$$\bar{a}_i = a_i / \sum_{i=1}^{m} a_i \qquad n_{\bar{a}} = \sum_{i=1}^{m} \bar{a}_i n_i = \Sigma a_i n_i / \Sigma a_i$$

$$\bar{\alpha}_i = \alpha_i / \sum_{i=1}^{m} \alpha_i \qquad n_{\bar{\alpha}} = \sum_{i=1}^{m} \bar{\alpha}_i n_i = \Sigma \alpha_i n_i / \Sigma \alpha_i$$

$$n_{(1)} = \min(n_1, \ldots, n_m)$$

Theorem 3.1 : Let $h(q_u;\underline{a})$ denote the true lower confidence bound for a series system of order m, and let $h(\tilde{q}_u;\underline{\alpha})$ be the estimated lower bound. Then the absolute difference, D, between the true and estimated confidence bounds for a series system of binomial components of order m satisfies:

$$(3.2)\quad D = |h(q_u;\underline{a}) - h(\tilde{q}_u;\underline{\alpha})| < \frac{s-s^{m+1}}{1-s} \quad \text{where } s = \frac{\lambda_u}{n_{(1)}}$$

and if

$$(3.3)\qquad n_{\bar{a}} = n_{\bar{\alpha}} \equiv \bar{n}$$

then

$$(3.4)\quad |h(q_u;\underline{a}) - h(\tilde{q}_u;\underline{\alpha})| < \frac{t^2}{2}\left(1-\frac{1}{m}\right)+\frac{t^3}{1-t}\left(1-t^{m-2}\right) \quad \text{where } t=\frac{\lambda_u}{\bar{n}}$$

Proof : By definition we obtain for the reliability of a series system

$$(3.5)\quad \prod_{i=1}^{m} h(\underline{p}) = \prod_{i=1}^{m} h(1-a_i q) = 1 - s_{1a}q + s_{2a}q^2 \pm \dots \pm s_{ma}q^m$$

where $s_{1a} = \sum_{i=1}^{m} a_i \quad s_{2a} = \sum_{i<j}^{m} a_i a_j \quad \dots \quad s_{ma} = \prod_{i=1}^{m} a_i$.

From (3.5),

$$(3.6)\quad D \le |s_{1a}q_a - s_{1\alpha}\tilde{q}_\alpha| + |s_{2a}q_a - s_{2\alpha}\tilde{q}_\alpha| + \dots + |s_{ma}q_a - s_{m\alpha}\tilde{q}_\alpha| .$$

Recalling (2.3) and (3.1), (3.6) becomes

$$(3.7)\quad D \le \lambda_u\left|\frac{\Sigma a_i}{\Sigma a_i n_i} - \frac{\Sigma \alpha_i}{\Sigma \alpha_i n_i}\right| + \lambda_u^2\left|\frac{\sum_{i<j} a_i a_j}{(\Sigma a_i n_i)^2} - \frac{\sum_{i<j}\alpha_i\alpha_j}{(\Sigma\alpha_i n_i)^2}\right| + \dots$$

$$\dots + \lambda_u^m\left|\frac{\Pi a_i}{(\Sigma a_i n_i)^m} - \frac{\Pi \alpha_i}{(\Sigma \alpha_i n_i)^m}\right| .$$

To establish (3.2), we note that if $a_i > 0$ for all i, then

$$(3.8)\qquad \frac{\sum_{i_1<\cdots<i_k} a_{i_1} a_{i_2} \cdots a_{i_k}}{(\Sigma a_i n_i)^k} \le \left(\frac{1}{n_{(1)}}\right)^k .$$

By (3.8) and the fact that for positive x and y, $|x-y| \le \max(x,y)$, expression (3.7) may be bounded above by

$$\sum_{k=1}^{m} \left(\frac{\lambda_u}{n_{(1)}}\right)^k = \frac{s-s^{m+1}}{1-s} \qquad \text{where } s = \frac{\lambda_u}{n_{(1)}} .$$

Assuming (3.3) we note that (3.7) becomes

$$(3.9)\quad \mathcal{D} \le \left(\frac{\lambda_u}{\bar{n}}\right)^2 \left| \sum_{i<j} \bar{a}_i \bar{a}_j - \sum_{i<j} \bar{\alpha}_i \bar{\alpha}_j \right| + \cdots + \left(\frac{\lambda_u}{\bar{n}}\right)^m \left| \prod_1^m \bar{a}_i - \prod_1^m \bar{\alpha}_i \right| .$$

Let $t = \lambda_u / \bar{n}$, then (3.9) becomes

$$(3.10)\quad \mathcal{D} \le t^2 \left| \sum_{i<j} \bar{a}_i \bar{a}_j - \sum_{i<j} \bar{\alpha}_i \bar{\alpha}_j \right| + \cdots + t^m \left| \Pi \bar{a}_i - \Pi \bar{\alpha}_i \right| .$$

Using the method of La Grange multipliers on the first term of (3.10), the maximum value of $\sum_{i<j} \bar{a}_i \bar{a}_j$, subject to the restriction that $\sum_{i=1}^{m} \bar{a}_i = 1$, is obtained at $a_j = 1/m$ for $j=1,2,\ldots,m$. Thus

$$(3.11)\quad \left| \sum_{i<j} \bar{a}_i \bar{a}_j - \sum_{i<j} \bar{\alpha}_i \bar{\alpha}_j \right| \le \max\left(\sum_{i<j} \bar{a}_i \bar{a}_j , \sum_{i<j} \bar{\alpha}_i \bar{\alpha}_j \right) \le \frac{m-1}{2m} .$$

The other differences between the corresponding symmetric polynomials are certainly less than unity. Therefore, assuming them to be unity and performing the geometric sum, $\sum_{i=3}^{m} t^i = (t^3 - t^{m+1})/(1-t)$, we obtain a bound on the remaining terms. Assuming (3.3), we obtain

$$\mathcal{D} \le t^2 \left(\frac{n-1}{2n}\right) + \frac{t^3 - t^{m+1}}{1-t} \qquad \text{thus establishing (3.4).} \ ||$$

For small $n_{(1)}$, the bound obtained from (3.2) is large, particularly if one or more failures have been observed. For example the absolute error bound obtained for the data in Examples 2.2 and 3.1 is .24. In practice, however, when a sampling scheme such as that given in Example 2.2 is used, there is adequate engineering knowledge behind the choice of the α_i so that errors of such large magnitude are not encountered. If little is known about the weighting equal component sample sizes are recommended.

Example 3.2: Consider a series system with equal sample sizes for each component:

component	a_i	n_i	α_i
1	1	20	1/2
2	1/100	20	1
3	1/100	20	1/4

Assume no failures, then for a 90% confidence level $\lambda_u = 2.3026$.

The actual error in computing the series system confidence bound is

$$|h(\tilde{q}_u;\underline{\alpha})-h(q_u;\underline{a})| = |.889-.885| = .004$$

For this example, the AO bound is .892, a difference of .007 from the correct bound. However, by Theorem 3.1 an absolute bound on the error would be

$$t^2\left(\frac{m-1}{2m}\right) + \frac{t^3-t^{m+1}}{1-t} = .0059$$

where $t = \dfrac{\lambda_u}{n} = \dfrac{2.3026}{20}$ and $m = 3$.

Obviously if the true bound were higher, say .887, then the AO would differ from the true by only .005 and the actual error would be only .002. However, the point to be made is that we often have more information than simply the structure and the sample size and when we do, it should be used.

It should be noted that if the order of the structure is increased to 20 the absolute error bound is still only .0080.

We now wish to extend this error determination to a parallel structure of m components. In general, we denote the reliability of a parallel structure of binomial components by:

$$h(\underline{p}) = 1 - \prod_{i=1}^{m} (1-p_i) .$$

Writing $h(\underline{p})$ in terms of q we obtain

$$h(\underline{p}) = h(q;\underline{a}) = 1- \prod_{i=1}^{m} a_i q.$$

Theorem 3.2: Let $h(q_u;\underline{a})$ denote the true lower confidence bound for $h(\underline{p})$, and let $h(\tilde{q}_u;\underline{\alpha})$ denote the estimated lower confidence bound for $h(\underline{p})$. We obtain

$$(3.12) \qquad \mathcal{D} = |h(q_u;\underline{a}) - h(\tilde{q}_u;\underline{\alpha})| \leq \frac{(\lambda_u/m)^m}{\prod_{i=1}^{m} n_i} .$$

Proof : (3.12) is proved by noting that

$$(3.13) \quad \mathcal{D} = |\prod_{i=1}^{m} a_i q_u - \prod_{i=1}^{m} \alpha_i \tilde{q}_u| = (\lambda_u)^m \left| \frac{\prod_{i=1}^{m} a_i}{(\Sigma a_i n_i)^m} - \frac{\prod_{i=1}^{m} \alpha_i}{(\Sigma \alpha_i n_i)^m} \right| .$$

For $a_i>0$, the quantity $P = \frac{\prod_{i=1}^{m} a_i}{(\Sigma a_i n_i)^m} = \exp\left[\ln \frac{\prod_{i=1}^{m} a_i}{(\Sigma a_i n_i)^m}\right]$ is maximized when $\Sigma \ln a_i - m \ln(\Sigma a_i n_i)$ is maximized. This maximum occurs at $a_j = (\sum_{i=1}^{m} a_i n_i)/mn_j$ for $j=1,2,\ldots,m$. Thus the maximum value of P is $1/(m^m \Pi n_i)$. Since each

term in the difference $\left| \frac{\Pi a_i}{(\Sigma a_i n_i)^m} - \frac{\Pi \alpha_i}{(\Sigma \alpha_i n_i)^m} \right|$ is positive, the absolute value of the difference cannot exceed $1/(m^m \Pi n_i)$. From this (3.13) is established:

$$\mathcal{D} \leq \frac{(\lambda_u/m)^m}{\prod_{i=1}^{m} n_i} \,. \,||$$

Example 3.3: Consider a parallel structure of order two where no failures occur in testing

component	a_i	n_i	α_i
1	1/2	40	1
2	1	20	1/10

For a 90% confidence bound $\lambda_u = 2.3026$. The 90% lower confidence bounds for system reliability are:

$$h(q_u;\underline{a}) = 1-q_u{}^2 \prod_{i=1}^{2} a_i = .99834$$

$$h(\tilde{q}_u;\underline{\alpha}) = 1-\tilde{q}_u{}^2 \prod_{i=1}^{2} \alpha_i = .99970$$

The actual difference between these bounds is 1.36×10^{-3}. From Theorem 3.2 the maximum possible error due to incorrectly choosing the weights is 1.66×10^{-3}.

The AO bound is .99829 with a difference from the correct bound of 0.5×10^{-4}.

We now consider a series parallel structure of order $m = \sum_{i=1}^{k} m_i$ which consists of k parallel structures in series; the i^{th} parallel structure has m_i components. Partition $\underline{p} = (p_1, p_2, \ldots, p_m)$ into $\underline{p} = (\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_k)$ where $\underline{p}_j = (p_{m_1+\ldots+m_{j-1}+1}, \ldots, p_{m_1+\ldots+m_j})$. The reliability of the system is given by $h(\underline{p}) = \prod_{i=1}^{k} h_i(\underline{p}_i)$ with $h_j(\underline{p}_j)$ being a parallel structure consisting of m_j components. Now let $q \equiv \max(q_1, \ldots, q_m)$ where again $q_i = a_i q$ for $i=1,\ldots,m$. The corresponding partition of $\underline{a}$ is $\underline{a} = (\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_k)$. Therefore expressing the reliability in terms of q we obtain

$$h(q;\underline{a}) = \prod_{i=1}^{k} h_i(q;\underline{a}_i) .$$

Therefore we establish the error on the lower confidence bound estimate for a series parallel system by

Theorem 3.3 : Let $h_i(q_u;\underline{a}_i)$ be the true lower confidence bound for the i^{th} $(i=1,2,\ldots,k)$ parallel structure of order m_i , and let $h_i(\tilde{q}_u;\underline{\alpha}_i)$ be the associated estimate of the lower confidence bound. Then an absolute bound on error in estimating the confidence bound for a series parallel system is given by:

$$(3.14) \quad \left| \prod_{i=1}^{k} h_i(q_u;\underline{a}_i) - \prod_{i=1}^{k} h_i(\tilde{q}_u;\underline{\alpha}_i) \right| \le \sum_{i=1}^{k} \varepsilon_i$$

where

$$(3.15)\qquad \varepsilon_i = \left(\frac{\lambda_u}{m_i}\right)^{m_i}\left(\frac{1}{\prod_{j=1}^{m_i} n_j}\right),$$

that is ε_i is the bound on error of the i^{th} parallel structure. The n_j are the sample sizes of the components made during the qualification tests.

<u>Proof</u> : By induction. For the case when k=1, see Theorem 3.2.

Let k=2. For convenience set

$f_i = h_i(q_u;\underline{a}_i)$ and $\delta_i = h_i(\tilde{q}_u;\underline{a}_i)$.

Then $|f_1 \cdot f_2 - \delta_1 \cdot \delta_2| = |f_1 \cdot f_2 - \delta_1 \cdot \delta_2 - \delta_2 \cdot f_1 + \delta_2 \cdot f_1|$

$$= |f_1(f_2 - \delta_2) + \delta_2(f_1 - \delta_1)| \le |f_1(f_2 - \delta_2)| + |\delta_2(f_1 - \delta_1)|$$

$$\le |f_2 - \delta_2| + |f_1 - \delta_1| = \varepsilon_1 + \varepsilon_2.$$

Assume expression (3.14) is true for k=n . It holds for k=n+1 since,

$$\left|\prod_{i=1}^{n+1} f_i - \prod_{i=1}^{n+1} \delta_i\right| = \left|\prod_{i=1}^{n+1} f_i - \prod_{i=1}^{n+1} \delta_i + f_{n+1}\prod_{i=1}^{n} \delta_i - f_{n+1}\prod_{i=1}^{n} \delta_i\right|$$

$$= \left|\prod_{i=1}^{n} \delta_i(f_{n+1} - \delta_{n+1}) + f_{n+1}\left(\prod_{i=1}^{n} f_i - \prod_{i=1}^{n} \delta_i\right)\right|$$

$$\le \left|\prod_{i=1}^{n} \delta_i(f_{n+1} - \delta_{n+1})\right| + \left|f_{n+1}\left(\prod_{i=1}^{n} f_i - \prod_{i=1}^{n} \delta_i\right)\right|$$

$$\le |f_{n+1} - \delta_{n+1}| + \left|\prod_{i=1}^{n} f_i - \prod_{i=1}^{n} \delta_i\right| \le \text{(by induction)}$$

$$|f_{n+1} - \delta_{n+1}| + \sum_{i=1}^{n} |f_i - \delta_i| = \sum_{i=1}^{n+1} |f_i - \delta_i| = \sum_{i=1}^{n+1} \varepsilon_i \;. \;||$$

Using a similar partitioning procedure for a parallel

series structure of order $m = \sum_{i=1}^{k} m_i$, then this system consists of k series structures in parallel, where the i^{th} series structure has order m_i . Then $\underline{p} = (\underline{p}_1, \ldots, \underline{p}_k)$, $h(\underline{p}) = 1 - \prod_{j=1}^{k} (1-h_j(\underline{p}_j))$ and $h(q;\underline{a}) = 1 - \prod_{j=1}^{k} (1-h_j(q;\underline{a}_j))$.

The maximum error made by estimating the lower confidence bound is obtained from

<u>Theorem 3.4</u> : Let $h(q_u;\underline{a}) = 1 - \prod_{j=1}^{k} (1-h_j(q_u;\underline{a}_j))$ be the true confidence bound on the reliability of a parallel series structure of order $m = \sum_{i=1}^{k} m_i$. And let $h(\tilde{q}_u;\underline{\alpha}) = 1 - \prod_{j=1}^{k} (1-h_j(\tilde{q}_u;\underline{\alpha}_j))$ be the associated estimate of this lower confidence bound. Then

$$B = |h(q_u;\underline{a}) - h(\tilde{q}_u;\underline{\alpha})| \leq \sum_{j=1}^{k} \tau_j \tag{3.16}$$

where $\tau_j = \dfrac{s_j - s_j^{m+1}}{1-s_j}$ and s_j is defined in Theorem 3.1 .

<u>Proof</u> : Expanding B we obtain

$$B = \left| \prod_{j=1}^{k} (1-h_j(q_u;\underline{a}_j)) - \prod_{j=1}^{k} (1-h_j(\tilde{q}_u;\underline{\alpha}_j)) \right| .$$

By Theorem 3.3,

$$B \leq \sum_{j=1}^{k} |(1-h_j(q_u;\underline{a}_j)) - (1-h_j(\tilde{q}_u;\underline{\alpha}_j))|$$

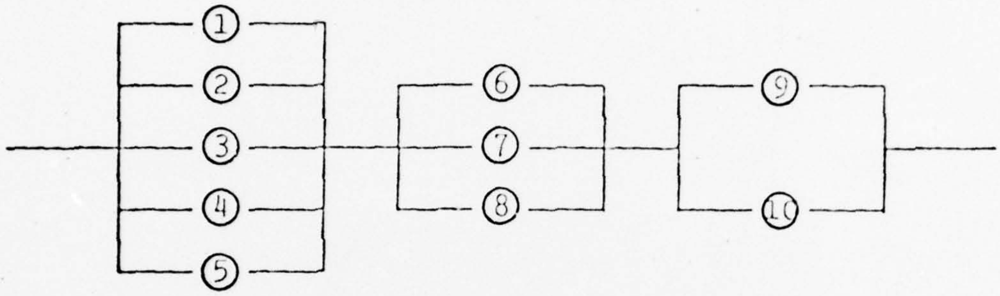
$$= \sum_{j=1}^{k} |h_j(q_u;\underline{a}_j) - h_j(\tilde{q}_u;\underline{\alpha}_j)|$$

where for the i^{th} branch of this parallel structure we define (by Theorem 3.1)

$$\tau_j \equiv |h_j(q_u;\underline{a}_j) - h_j(\tilde{q}_u;\underline{\alpha}_j)| = \frac{s_j - s_j^{m+1}}{1-s_j} \quad .$$

Thus $B \leq \sum_{j=1}^{k} \tau_j$. ||

Example 3.4 : Consider the following structure.



component	a_i	n_i	α_i
1	1/4	100	1
2	1/4	100	1
3	1/4	100	1
4	1/4	100	1
5	1/4	100	1
6	1	150	3/4
7	1	150	3/4
8	1	150	3/4
9	3/4	250	1/4
10	3/4	250	1/4

We assume $\Sigma X_i = 1$ and we desire an 80% confidence level. This will yield $\lambda_u = 2.99$. Simple calculations show us

$$\Sigma a_i n_i = 950 \qquad q_u = .00315$$

$$\Sigma \alpha_i n_i = 962.5 \qquad \tilde{q}_u = .00311$$

$$h(q_u;\underline{a}) = (1-(.25q_u)^5)(1-q_u^3)(1-(.75q_u)^2) = .999994$$

$$h(\tilde{q}_u;\underline{\alpha}) = (1-\tilde{q}_u^5)(1-(.75\tilde{q}_u)^3)(1-(.25\tilde{q}_u)^2) = .999999 \ .$$

We realize an actual error of 5.38×10^{-6}, and based on Theorem 3.3, a maximum possible error of 3.6×10^{-5}.

4. Exponential Component Failure Distributions

We now examine components whose life lengths are known to be exponential. If the qualification test time for the i^{th} component is T_i , where $i=1,2,\ldots,m$, then the number of failures during the interval $[0,T_i]$ follows a Poisson distribution with mean $\lambda_i T_i$. Define

(4.1) $\lambda_i = a_i\lambda$ where $\lambda = \max_{i=1}^{m} \lambda_i$ and $0<a_i\leq1$.

Let X_i denote the number of observed failures during $[0,T_i]$, then $X_i \sim P(a_i\lambda T_i)$. If all m components are independent, $\Sigma X_i \sim P(\lambda\Sigma a_i T_i)$.

The upper confidence bound obtained here is completely analogous to the bound obtained in (2.3), i.e. the upper $100\beta\%$ confidence bound on λ, say $\hat{\lambda}$, is

$$\hat{\lambda} = \lambda_u/\Sigma a_i T_i \ .$$

Reliability at time t, $R(t;\lambda_1,\ldots,\lambda_m)$, may be written as $h(\lambda t;\underline{a})$ where λ and $\underline{a}$ are defined in (4.1). It follows that a lower $100\beta\%$ confidence bound on system reliability is given by $h(\hat{\lambda}t;\underline{a})$ where $\underline{a}$ is the vector of true weights.

5. Sensitivity of Confidence Bounds to Assumed Weights

When the true weights $\underline{a}$ are estimated by $\underline{\alpha}$ then the estimated $100\,\beta\%$ lower confidence bound for system reliability becomes

(5.1) $$h(\tilde{\lambda}t;\underline{\alpha}) \quad \text{where } \tilde{\lambda} = \lambda_u/\Sigma\alpha_i T_i \ .$$

For the exponential series case we measure error by means of the ratio:

$$(5.2) \qquad R \equiv \frac{h(\hat{\lambda}t;\underline{a})}{h(\tilde{\lambda}t;\underline{\alpha})}$$

We will show that in the case of equal component test times for series systems, the above ratio is identically one. That is, the estimate is equal to the true confidence bound for any weighting $\underline{\alpha}$.

<u>Theorem 5.1</u> : For a series system of order m,

$$(5.3) \quad \exp -\lambda_u t\left(\frac{1}{T_{(1)}} - \frac{1}{T_{(m)}}\right) \leq R \leq \exp \lambda_u t\left(\frac{1}{T_{(1)}} - \frac{1}{T_{(m)}}\right)$$

where $T_{(1)} = \min(T_1,\ldots,T_m)$ and $T_{(m)} = \max(T_1,\ldots,T_m)$.

If for $i=1,\ldots,m$, $T_i \equiv T$ then .

$$(5.4) \qquad h(\hat{\lambda}t;\underline{a}) \equiv h(\tilde{\lambda}t;\underline{\alpha}) = \exp \frac{-\lambda_u t}{T} .$$

<u>Proof</u> : By expanding the ratio in (5.2) we observe

$$R \equiv \frac{h(\hat{\lambda}t;\underline{a})}{h(\tilde{\lambda}t;\underline{\alpha})} = \frac{\exp \dfrac{-\lambda_u t}{\Sigma a_i T_i}\,\Sigma a_i}{\exp \dfrac{-\lambda_u t}{\Sigma \alpha_i T_i}\,\Sigma \alpha_i} = \exp -\lambda_u t\left|\frac{\Sigma a_i}{\Sigma a_i T_i} - \frac{\Sigma \alpha_i}{\Sigma \alpha_i T_i}\right| .$$

It is simple to see that

$$\frac{1}{T_{(m)}} \leq \frac{\Sigma a_i}{\Sigma a_i T_i} \leq \frac{1}{T_{(1)}}$$

thus

$$(5.5) \quad -\left(\frac{1}{T_{(1)}} - \frac{1}{T_{(m)}}\right) \leq \left|\frac{\Sigma a_i}{\Sigma a_i T_i} - \frac{\Sigma \alpha_i}{\Sigma \alpha_i T_i}\right| \leq \left(\frac{1}{T_{(1)}} - \frac{1}{T_{(m)}}\right) .$$

Therefore based on (5.5)

$$\exp -\lambda_u t\left(\frac{1}{T_{(1)}} - \frac{1}{T_{(m)}}\right) \leq R \leq \exp \lambda_u t\left(\frac{1}{T_{(1)}} - \frac{1}{T_{(m)}}\right) .$$

If for all i, $T_i \equiv T$, then the inequality expressed in (5.5)

becomes

$$\left|\frac{\Sigma a_i}{\Sigma a_i T_i} - \frac{\Sigma \alpha_i}{\Sigma \alpha_i T_i}\right| \leq \left|\frac{1}{T_{(1)}} - \frac{1}{T_{(m)}}\right| \equiv 0$$

therefore the ratio of (5.2) is identically one. That is

$$h(\hat{\lambda} t;\underline{a}) = \exp \frac{-\lambda_u t}{T\Sigma a_i}\Sigma a_i = \exp \frac{-\lambda_u t}{T\Sigma \alpha_i}\Sigma \alpha_i = h(\tilde{\lambda} t;\underline{\alpha})$$

$$\equiv \exp -\lambda_u t/T \ . \quad ||$$

Lieberman and Ross [3] have derived a method for obtaining confidence bounds for series systems whose components have an exponential life distribution. The test statistic used in their method is based on the sum of simulated system failure times. For Type I censoring the Lieberman-Ross technique will, in general, not utilize all of the test data in the calculation of their confidence bound. In the case of no observed failures the Lieberman-Ross method is not applicable. As shown by the following example, the procedure we propose is not hampered by an absence of qualification test failures.

Example 5.1: Assume that we have twenty components in series and that no failures have been experienced during testing. As is often encountered in practice, the test times are not all equal.

component	a_i	T_i
1	1	10
2 through 20	1/10	100

$\sum_{i=1}^{20} a_i T_i = 200$ and $\sum_{i=1}^{20} a_i = 2.9$. At the 80% confidence level $\lambda_u = 1.61$. Let t=1; then

$$h(\hat{\lambda}t;\underline{a}) = \exp-\left[\frac{1.61}{200}(2.9)\right] = .977 .$$

The AO bound is .852. The PA bound is .839.

Assume that the α_i were not chosen correctly but were chosen according to one of the cases given below.

component	$\underline{\alpha_i}^{(1)}$	$\underline{\alpha_i}^{(2)}$	$\underline{\alpha_i}^{(3)}$	$\underline{\alpha_i}^{(4)}$	$\underline{T_i}$
1	1	1	1	1/10	10
2 through 20	1/100	1/2	1	1	100
$h(\tilde{\lambda}t;\underline{\alpha})$	.936	.983	.983	.984	

The maximum difference from the true bound is .041. Based on Theorem 5.1 the ratio is bounded by

$$.865 \le R \le 1.156.$$

Thus we know that regardless of the weights the 80% bound must be greater than .851 (the AO bound is .852). Again, the point to be made here is that we often have more information than simply the structure and the sample size and when we do, it should be used. For this example, using the ratio of Theorem 5.1, it is possible to show that regardless of the weighting the true bound is at least as large as the AO bound (to two significant figures).

In the case of equal test times, say 100, the weighting method bounds are exact and equal to .984. The AO bound is also .984.

Theorem 5.2 : For a parallel system of m exponential components with $0<\lambda_i t<.1$ for $i=1,2,\ldots,m$, the difference

(5.6) $$\mathcal{D} = |h(\hat{\lambda}t;\underline{a}) - h(\tilde{\lambda}t;\underline{\alpha})|$$

is approximately bounded above by

(5.7) $$\frac{(\lambda_u t/m)^m}{\prod_{i=1}^{m} T_i} .$$

Proof : For a parallel system of exponential components

$$R(t) = 1 - \prod_{i=1}^{m}\left(1-e^{-\lambda_i t}\right).$$

If $|\lambda t| < .1$ then $\lambda t \approx 1-e^{-\lambda t}$. Therefore

$$\mathcal{D} \approx \left|\prod_{i=1}^{m} \hat{\lambda}ta_i - \prod_{i=1}^{m} \tilde{\lambda}t\alpha_i\right| = (\lambda_u t)^m \left|\frac{\Pi a_i}{(\Sigma a_i T_i)^m} - \frac{\Pi \alpha_i}{(\Sigma \alpha_i T_i)^m}\right| .$$

Then by Theorem 3.2, $\mathcal{D}$ is approximately less than or equal to

$$\frac{(\lambda_u t/m)^m}{\prod_{i=1}^{m} T_i} . \quad ||$$

6. Conclusion

The weighting method developed in this paper allows engineering knowledge to be used in a very simple and feasible manner. If little is known about the weighting factors, then we know that for component sample sizes of as small as 20,

if there are no failures, the absolute error bound for a series system of order "∞" is still less than .008. In practice the actual errors induced due to incorrect choice of the weighting factors is much less than the absolute bounds.

Sensitivity studies show that qualification test sample sizes (test times) increase, the effects of the weights on the estimated confidence bound decreases. If little is known about the weighting factors, the bound on the maximum possible error induced by different weights may be reduced significantly by imposing equal sample sizes (test times) for the components during testing. Moreover, under the assumption of equivalent component test times in the commonly encountered case of exponential series systems, the confidence bound obtained is exact. The advantages of the weighting method proposed here lie in the simplicity of the calculations, the applicability to <u>any</u> coherent structure when few or no failures occur, the ability to use in an uncomplicated fashion engineering knowledge to compensate for small sample sizes (test times), and, for larger sample sizes, the insensitivity to the choice of weighting factors.

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ATTACHMENT B

UNDER REVISION

PROBLEMS OF ESTIMATION FOR A DECREASING FAILURE RATE DISTRIBUTION APPLIED TO RELIABILITY *

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0. Abstract

In this paper a mixed exponential distribution is studied. This is a two-parameter family, with a decreasing failure rate, which is sometimes called the Lomax distribution or the Pareto distribution of the second kind. The properties of this model are examined along with their implications in determining reliability assessment. Simple techniques are derived which yield sufficient conditions for obtaining the solution of the maximum likelihood equations when one or both of the parameters are unknown, even when the data is highly truncated or censored. More importantly, computational techniques are obtained and tested which affect their efficient calculation. Censored data is the usual occurrence when life testing many types of equipment because the cost makes complete samples virtually unobtainable.

The results of this paper are applied to censored data, (obtained from actual testing of flight control electronic packages), in which failure observations are sparse.

Key Words

Reliability
Decreasing Failure Rate
Mixed Exponential
Censored Sample
Maximum Likelihood Estimation
Burn-in

1. Introduction

There have been only a few parametric models extensively examined for application to reliability; these include the exponential distribution of Epstein-Sobel [6], the Weibull distribution [14], and the fatigue model of Birnbaum-Saunders [4]. The one most widely utilized for electronic components has been the exponential model, not only because of its simple and intuitive properties but also because of the extent of the estimation and sampling procedures which have been developed from the theory.

One concomitant development has been the investigations of Barlow and Proschan [2] concerning models for life distributions which are known only to have monotone increasing (or decreasing) hazard rates. Of course, the exponential family serves as the boundary between these classes of distributions with monotone hazard rate and consequently serves as an extremum for the results of either case.

One of the early discoveries was that mixtures of exponentially distributed random variables have a decreasing failure rate, see [11]. Thus any two groups of components with constant, but different, failure rates would, if mixed and sampled at random, exhibit a decreasing failure rate. As a consequence, the family of life lengths with decreasing failure rate certainly arises

in practice and particular subsets of this family could be of great utility for specific applications, see e.g. Cozzolino [5]. We examine one such model with shape and scale parameters, call them α and β respectively, which is based upon a particular mixture of exponential distributions. This family was introduced by Afanas'ev [1] and later by Lomax [10] as a generalization of a Pareto distribution.

Kulldorff and Vännman [9] and Vännman [13] have studied a variant of this mixed exponential model containing a location parameter. They obtained a best linear unbiased estimate of the scale parameter assuming that the shape parameter, call it α, was known and in a region restricted so that both the mean and the variance exist, namely $\alpha > 2$. When this restriction of $\alpha > 2$ cannot be met an estimate based on a few order statistics, which are optimally spaced, is claimed to be an asymptotically best linear unbiased estimate and tables of the weights as functions of the number of spacings are provided. In all cases, the shape parameter was assumed known and the sample was either complete or type II censored. It is contended that BLUE estimates of the shape parameter are not attainable.

Harris and Singpurwalla [7] examined the method of moments as an estimation procedure for this same model but again with the shape parameter restricted to $\alpha > 2$ and with a complete sample.

In both papers [9] and [7] , it is stated that maximum likelihood estimates are difficult to obtain. In a later paper Harris and Singpurwalla [8] exhibit the maximum likelihood equations for complete samples.

In this paper the maximum likelihood estimates for both the shape and scale parameters are obtained, jointly and separately, with simple sufficient conditions given for their existence. These estimates are derived for censored data (and *a fortori* for complete samples) even with a paucity of failure observations, namely one.

The existence conditions obtained here for the maximum likelihood estimates apply even to the case where the variance and possibly the mean do not exist: $0<\alpha<2$. Moreover, the estimates of the shape parameter α which have been obtained from actual data indicate that this region $0<\alpha<2$ is important because all the estimates obtained of α have been less than unity.

2. The Model

We postulate that the underlying process which determines the length of life of the component under consideration is the following: The quality of construction determines a level of resistance to stress which the component can tolerate. The service environment provides shocks of varying magnitude to the component and failure takes place when for the first time the stress from an environmentally induced shock exceeds the strength of the component.

If the time between shocks of any magnitude is exponentially distributed with a mean depending upon that magnitude then the life length of each component will be exponentially distributed with a failure rate which is determined by the quality of assembly. It follows that each component has a constant failure rate but that the variability in manufacture and inspection techniques forces some components to be extremely good while a few others are bad and most are in between.

Let X_λ be the life length of a component in such a service environment, with a constant failure rate λ which is unknown. The variability of manufacture determines various percentages of the λ-values and this variability can be described by some distribution, say G.

Let T be the life length of one of the components which is selected at random from the population of manufactured components. We denote the reliability of this component by R and we have

$$R(t) = P[T > t] \qquad \text{for} \quad t > 0 .$$

Let Λ be the random variable which has distribution G. We can write

$$R(t) = E_\Lambda P[X_\lambda > t \mid \Lambda = \lambda] = \int_0^\infty e^{-\lambda t} dG(\lambda) . \qquad (1)$$

Because of having a form which can fit a wide variety of practical situations when both scale and shape parameters are disposable, it is assumed that G is a gamma distribution, i.e.

for some $\alpha>0$, $\beta>0$

$$g(\lambda) = \frac{\lambda^{\alpha-1} e^{-\lambda/\beta}}{\Gamma(\alpha)\ \beta^{\alpha}} \qquad \text{for } \lambda > 0.$$

That this assumption is robust, even when mixing as few as five equally wieghted λ's, has been shown by recent work of Sunjata in an unpublished thesis [12]. It follows from equation (1) that the reliability function is

$$R(t) = \frac{1}{(1+t\beta)^{\alpha}} = e^{-\alpha\ \ell n(1+t\beta)} \quad . \tag{2}$$

The failure rate can be shown to be

$$q(t) = \frac{\alpha\beta}{1+t\beta} \quad , \tag{3}$$

which in accord with the result of Barlow and Proschan [2], is a decreasing function of $t > 0$.

A similar derivation and discussion has been given earlier by Cozzolino [5] and is included here for completeness, as well as for what insight it may provide into the physical meaning of the parameters.

For this family, since $\ell n(1+t\beta) < t\beta$ for all $\beta > 0$ and $t > 0$

$$R(t) > e^{-\alpha\beta t} \tag{4}$$

This simple relationship provides a basis for comparison subsequently with the exponential model.

It can be shown that

$$\begin{aligned} E[T] &= [\beta(\alpha-1)]^{-1} && \text{for } \alpha > 1 \\ Var[T] &= \alpha[\beta^2(\alpha-1)^2(\alpha-2)]^{-1} && \text{for } \alpha > 2 \end{aligned} \tag{5}$$

These formulae have been given previously in numerous places. When it exists, the standard deviation exceeds the mean as it will for all DFR distributions. For example, if $\alpha = 4$, then we see the standard deviation exceeds the mean be a factor of $\sqrt{2}$.

Of course, alternative parameterizations can be made for this same model of mixtures of exponentially lived components. We now mention such an alternative parameterization for later comparison and discussion. Let $\alpha = \gamma/\beta$ in equation (2) and we obtain the distribution

$$R(t) = e^{-(\gamma/\beta)\ell n(1+t\beta)} \quad \text{for } t > 0 , \tag{6}$$

with hazard and hazard rate, respectively, given by

$$Q(t) = \gamma/\beta \int_0^{\beta t} \frac{dx}{1+x} , \quad q(t) = \frac{\gamma}{1+\beta t} \quad \text{for } t > 0 .$$

The reason for this parameterization is that as $\beta \to 0$ we are provided with a meaningful limiting distribution, namely the exponential, since

$$\lim_{\beta \to 0} R(t) = \exp\left\{-\gamma \lim_{\beta \to 0} \frac{\ell n(1+t\beta)}{\beta}\right\} = e^{-\gamma t} .$$

However, in the formulation given in equation (7), β is no longer a scale parameter.

3. <u>Properties of the Model</u>

We now turn to a brief examination of some of the properties of this family of distributions as expressed in the two alternate parameterizations. Let us introduce the notation for a random life length T :

$$T \sim J_1(\alpha,\beta) \quad \text{when} \quad P[T > t] = e^{-\alpha \ell n(1+t\beta)} \quad \text{for } t > 0 , \quad (7)$$

$$T \sim J_2(\gamma,\beta) \quad \text{when} \quad P[T > t] = e^{-(\gamma/\beta)\ell n(1+t\beta)} \quad \text{for } t > 0 . \quad (8)$$

The first property, which it has in common with the Weibull distribution (and a fortiori the exponential), is contained in

Theorem 1: If T_i is a life length with survival distribution $J_1(a_i,\beta)$, as defined in equation (7), for $i = 1,\ldots,n$,

then a series system of n such independent components will have a survival distribution $J_1(\alpha_1+\dots+\alpha_n,\beta)$ in the same family.

Proof: Simply consider

$$P\left[\min(T_1,\dots,T_n) > t\right] = \prod_{i=1}^{n} P\left[T_i > t\right] = \frac{1}{(1+t\beta)^{\alpha_1+\dots+\alpha_n}} \ . \ ||$$

Note that the remark is the same even if we use the second parameterization of the model, namely,

Corollary 1: If $T_i \sim J_2(\gamma_i,\beta)$ for $i = 1,\dots,n$ are independent observations as defined in equation (8) then

$$\min_{i=1}^{n} T_i \sim J_2(\gamma_1+\dots+\gamma_n,\beta) \ .$$

The importance of this property is in the case of determining the time to first failure within a fleet of similar systems which have possibly different shape parameters. From this result the appropriate fleet reliability calculations can be made.

The next property, which we consider to be even more significant, is that a "burn-in" test of a component will yield a residual life which is also in the same family. The residual life T_h of a component is defined to be the life remaining after time h given that the component is alive at time h , namely

$$T_h = \left[T-h \mid T > h\right] \ .$$

This property seems to be shared only with the exponential among

common parametric families of life distributions. The result is

Theorem 2: If q_θ is the hazard rate for a random variable T with parameter $\theta\varepsilon\Omega$ then the conditions for residual life T_h to be in the same family is that for any $h > 0, \theta\varepsilon\Omega$, there exists $\omega\varepsilon\Omega$ such that

$$q_\theta(t+h) = q_\omega(t) \qquad \text{for all } t \geq 0.$$

Thus we have

Corollary 2: A burn-in for h units of time on a component with initial life $T \sim J_1(\alpha,\beta)$ will yield a residual life $T_h \sim J_1[\alpha,\beta/(1+h\beta)]$. If $T \sim J_2(\gamma,\beta)$ then $T_h \sim J_2\left(\frac{\gamma}{1+h\beta}, \frac{\beta}{1+h\beta}\right)$.

It follows that this life length model is "used better than new" or "new worse than used" in the sense that we have stochastic inequality between a new component and one that has been burned in, namely

$$T \overset{st}{\leq} T_h \qquad \text{for all} \quad h > 0.$$

An important consequence of this property is that one can calculate the value of the increased reliability attained by burn-in procedures as compared with the cost of conducting them. It has long been the practice to burn in electronic components based on intuitive ideas of "infant mortality" in order to provide reasonable assurance of having detected all defectively assembled units. This model, whenever it is applicable, makes possible an economic analysis. A variation of this result has been discussed in [3].

4. A Comparison with Exponential Using Real Data

Data has been accumulating for years in the assessment of the reliability of electronic equipment for which there was no adequate statistical model. The following difficulties were recognized by practitioners: 1. The assumption of constant or increasing failure rate seemed to be incorrect. 2. However, the design of this electronic equipment indicated that individual items should exhibit a constant failure rate. A mixed exponential life distribution accounts for both the design knowledge and the observed life lengths. Maximum likelihood procedures allow for joint estimation of the parameters of this distribution in the most commonly encountered situation where complete data is not available.

We now give some actual data sets from two different lots of flight control electronic packages which illustrate these points. Each package has recorded, in minutes, either a failure time or an alive time. An alive time is sometimes called a "run-out" and is the time the life test was terminated with the package still functioning.

First Data Set

Failure times: 1, 8, 10

Alive times: 59, 72, 76, 113, 117, 124, 145, 149, 153, 182, 320.

Second Data Set

Failure times: 37, 53

Alive times: 60, 64, 66, 70, 72, 96, 123.

If we assume that the data are observations from an exponential distribution (constant failure rate λ) then using the total

life statistic, we have the estimates of reliability given in the left hand side of the table. If we assume that the data are observations from the mixed exponential distribution of equation (2) then using estimation techniques derived subsequently in this paper we have the estimates for reliability given in the right hand side of the table.

	Exponential est. of reliability		Mixed exponential est. of reliability	
time t in min.	Set 1 $\hat{R}_1(t)$	Set 2 $\hat{R}_2(t)$	Set 1 $\hat{R}_1(t)$	Set 2 $\hat{R}_2(t)$
6	.988	.981	.915	.976
10	.980	.969	.896	.961
30	.943	.911	.855	.896
50	.906	.856	.836	.843
100	.821	--	.810	--
130	.774	--	.801	--
$\hat{\lambda}$:	.00017	.00312	$\hat{\alpha}$: .0453	.420
			$\hat{\beta}$: 1.03	.01

Looking at the data from the two sets we would expect that at least for the first fifty minutes the reliability estimate for the second set of data would be <u>higher</u> then the reliability estimate for the first set of data, because in the first set 3 failures out of 14 trials have occurred in the first ten minutes while in the second set only 1 failure out of 9 trials has occurred in the first fifty minutes. However, under the exponential assumption the reliability estimates for the first data set are consistently higher.

Note that the mixed exponential estimates are more consistent with what the data show; that is, for at least the first 50 minutes we expect the reliability estimate for the second set of data to be higher than the reliability estimate for the first set of data. Beyond this time, however, say at 100 minutes, the data indicate that the reliability estimate from the first set of data should be higher than the reliability estimate from the second set of data. Using mixed exponential estimates this is the case.

A statistical test to determine whether the data require a constant or decreasing failure rate was run on the data from Sets 1 and 2. For data Set 1 we reject constant failure rate in favor of decreasing failure rate at the .10 level. For data Set 2 we cannot reject the constant failure rate assumption. In this case, however, the constant failure rate estimates for reliability and the mixed exponential estimates for reliability are close. For data Set 2 one should not estimate reliability much beyond about 70 minutes since we do not have data to support those estimates.

5. Estimation of Parameters with Censored Data

Let us assume throughout this section that we are given $t_1,\ldots,t_k$ as observed times of failure while $t_{k+1},\ldots,t_n$ are observed alive-times both obtained from a $J_1(\alpha,\beta)$ life distribution with $1 \leq k \leq n$. We define two functions for $x > 0$.

$$S_1(x) = \sum_{i=1}^{n} \ln(1+t_i x) \ , \quad S_2(x) = \frac{1}{k}\sum_{i=1}^{k}(1+t_i x)^{-1} \tag{9}$$

A result on the maximum likelihood estimation (m.l.e.) of the unknown parameters is now given which utilizes data of this type.

Theorem 3: Under the assumptions and conditions given

(i) When $\beta > 0$ is known, there exists a unique m.l.e. of α, say $\hat{\alpha}$, given explicitly by

$$\hat{\alpha} = k/S_1(\beta) .$$

(ii) When $\alpha > 0$ is known, there exists a unique m.l.e. of β, say $\hat{\beta}$, given explicitly by

$$\hat{\beta} = A^{-1}(0)$$

where A is the monotone decreasing function defined by

$$A(x) = kS_2(x) - \alpha x S_1'(x) \quad \text{for} \quad x > 0$$

with primes denoting derivatives.

(iii) When α, β are both unknown, the m.l.e. of β, say $\hat{\beta}$, is given implicitly, when it exists positively and finitely, by

$$\hat{\beta} = B^{-1}(0)$$

where B is the function defined by

$$B(x) = \frac{S_2(x)}{x} - \frac{S_1'(x)}{S_1(x)} \quad \text{for} \quad x > 0$$

and the m.l.e. of α, say $\hat{\alpha}$, is given explicitly by

$$\hat{\alpha} = k/S_1(\hat{\beta}) .$$

Proof: As a result of the testing assumed, we have observed the events $[T_i = t_i]$ for $i = 1,\ldots,k$ and $[T_i > t_i]$ for $i = k+1,\ldots,n$. Then by definition, the log-likelihood L is given by

$$e^{L} = \prod_{i=1}^{k} f(t_i) \prod_{j=k+1}^{n} R(t_j) .$$

Substituting and taking logarithms we find

$$L = k \ln(\alpha\beta) - \alpha \sum_{i=1}^{n} \ln(1+t_i\beta) - \sum_{j=1}^{k} \ln(1+t_j\beta) .$$

In order to find the m.l.e.'s we consider the partial derivatives, which after substitution from (9) , are

$$\frac{\partial L}{\partial \alpha} = \frac{k}{\alpha} - S_1(\beta) , \tag{10}$$

$$\frac{\partial L}{\partial \beta} = -\alpha S_1'(\beta) + \frac{k}{\beta} S_2(\beta) . \tag{11}$$

Thus if $\beta > 0$ were known, we obtain $\hat{\alpha}$ from (10). This proves (i).

Correspondingly, if α were known, the m.l.e. of β , say $\hat{\beta}$, is obtained from $A(\beta) \equiv \beta \frac{\partial L}{\partial \beta} = 0$. So that $\hat{\beta} = A^{-1}(0)$. Note that A is a decreasing function of β which has $A(0) = k$, $A(\infty) = -n\alpha$. Thus there always exists exactly one solution $\hat{\beta}$ for any $\alpha > 0$ whenever $1 \le k \le n$. This proves (ii).

If α and β are both unknown then in order to obtain the maximum likelihood estimates jointly we must solve simultaneously equations (10) and (11). Substituting the solution for $\hat{\alpha}$ from equation (10) into (11) and dividing through by $k \ge 1$ yields $\hat{\beta}$,

as the solution, when it exists positively and finitely, of the equation $B(x) = 0$ where B has been defined previously. This proves (iii). ||

When both parameters are unknown in the $J_1(\alpha,\beta)$ distribution conditions are needed on the sample to insure that the maximum likelihood estimators α, β both exist.

Lemma 1: A necessary and sufficient condition that no m.l.e.'s exist for a (censored) sample from the $J_1(\alpha, \beta)$ distribution is that the inequality

$$\frac{1}{k}\sum_{i=1}^{k}\tau_i \sum_{j=1}^{n}\tau_j + \sum_{j=1}^{n}\ln(1-\tau_j)\cdot\frac{1}{k}\sum_{i=1}^{k}\tau_i(1-\tau_j) > \sum_{i=1}^{n}\tau_i{}^2 \tag{12}$$

be satisfied for all $x > 0$, where

$$\tau_i(x) = \frac{t_i x}{1+t_i x} \qquad i = 1, \cdots, n . \tag{13}$$

Proof: From (iii) we see $\hat{\beta}$ exists positively iff $\hat{\beta}$ is a zero of $B(x)$ on $(0,\infty)$ iff it is a zero of the function $C(x) = -xS_1(x)B(x)$ on the same interval since $xS_1(x)$ has no zeros there. Since $C = xS_1' - S_1S_2$, we see

$$C' = xS_1'' + S_1'(1-S_2) - S_2'S_1 \tag{14}$$

where from (9) we have

$$S_1'(x) = \sum_1^n \frac{t_i}{1+t_i x}, \quad S_1''(x) = \sum_1^n \frac{-t_i^2}{(1+t_i x)^2}, \quad S_2'(x) = \frac{1}{k}\sum_1^k \frac{-t_i}{(1+t_i x)^2}.$$

But notice that $C(0) = 0$ and since

$$\lim_{x\to\infty} xS_1''(x) = n, \quad \lim_{\infty} S_1 S_2 = \sum_{j=1}^n \frac{1}{k}\sum_{i=1}^k \lim_{x\to\infty} \frac{\ell n(1+t_j x)}{1+t_i x} = 0.$$

It follows that $C(\infty) = n$. It is clear that if $C'(x) > 0$ for all $x > 0$ then the extremes occur at $\hat{\beta} = 0$, $\hat{\alpha} = \infty$. Thus the m.l.e.'s do not exist, but since the sign of $C'(x)$ is the same as that of $xC'(x)$ for $x > 0$, by making the definitions in (13), we see a NASC that $C'(x) > 0$ is the inequality of equation (12). ||

Theorem 4: The inequality for $1 \leq k \leq n$

$$\frac{2}{k}\sum_{i=1}^k t_i \sum_{j=1}^n t_j < \sum_{j=1}^n t_j^2 \tag{15}$$

is a sufficient condition which a (censored) sample from a $J_1(\alpha,\beta)$ distribution must satisfy in order that maximum likelihood estimators of both parameters exist both positively and finitely.

Proof: To see that (15) implies that $\hat{\beta} > 0$ must exist, we note by continuity of C since $C'(0) = 0$, that it is sufficient to show that there exists an $x > 0$ for which $C''(x) < 0$. From equation (13) we find

$$C'(x) = \sum_{i=1}^n \frac{-t_i^2 x}{(1+t_i x)^2} + \frac{1}{k}\sum_{i=1}^k \frac{t_i x}{1+t_i x}\sum_{j=1}^n \frac{t_j}{1+t_j x} + \sum_{i=1}^n \frac{n(1+t_i x)}{k}\sum_1^k \frac{t_i}{(1+t_i x)^2}$$

We examine $\lim_{x \to 0} \frac{C'(x)}{x}$, making use of the fact that $\lim_{x \to 0} \frac{n(1+tx)}{x} = t$ and we use (15) to insure $C''(x) < 0$ for x sufficiently small.

We now examine the joint estimation problem for the alternatively parameterized model $J_2(\alpha,\beta)$ of equation (8) in the situation assumed heretofore, namely in the case we have both failure times and censored life times. Since the transformation $(\alpha,\beta) \to (\alpha/\beta,\beta)$ is one-to-one for $\alpha,\beta > 0$, the maximum likelihood estimators of γ,β are immediately obtained. One must check that at the boundary, $\beta = 0$, the maximum likelihood estimator $\hat{\gamma}$ reduces to the usual estimate obtained in the case of constant failure rate. But this can be done straightforwardly.

We state formally

Theorem 5: Under the assumptions and conditions given for a sample from a $J_2(\gamma,\beta)$ distribution when both parameters are unknown the m.l.e. of β is implicitly given, when it exists by $\hat{\beta} = B^{-1}(0)$ where B was defined in Theorem 3, and the m.l.e of γ is given by $\gamma = \frac{\hat{\alpha}}{\hat{\beta}}$. But in the case of $\hat{\beta} = 0$ we have $\hat{\gamma} = k/\sum_1^n t_i$ which is the usual estimate of a constant failure rate using the total life statistic.

6. Computational Considerations

The question which now arises is: what kinds of samples will satisfy condition (15)? If $k = n$ we see (15) is equivalent with

$$\left(\frac{1}{n}\sum_1^n t_i\right)^2 < \frac{1}{n}\sum_1^n t_i^{\,2} - \left(\frac{1}{n}\sum_1^n t_i\right)^2$$

from which we have the

Remark: A complete sample of failure times will satisfy (15) iff the sample standard deviation exceeds the sample mean.

Of course from equation (5) and the comment following we see that for the $J_1(\alpha,\beta)$ parameterization, the standard deviation does exceed the mean for those values of the parameters where the mean and standard deviation exist. Thus for all complete samples which are sufficiently large, this requirement should be satisfied.

Remark: A sample with $k < n$ failure times and the remaining $n-k$ observations truncated at t_0 will satisfy (15) if

$$t_0 > \eta_1\left[1 + \sqrt{\frac{2k}{n-k} + 1}\,\right] \simeq \eta_1 \frac{2n-k}{n-k} \quad \text{for } n \text{ large} \qquad (16)$$

where $\eta_1 = (t_1+\ldots+t_k)/k$ is the average failure time.

Proof: To see this note always $t_1{}^2+\ldots+t_n{}^2 > (n-k)t_0{}^2$ and thus we are assured that (15) must hold if

$$t_0{}^2 > 2t_0\eta_1 + \frac{2k}{n-k}(\eta_1)^2 .$$

By the quadratic formula this is equivalent with equation (16), with the second expression following from the first two terms of the binomial expansion. ||

In the calculation of $\hat{\beta}$ the equation, $C(\beta) = 0$, must be solved where $C(\beta) = \beta S_1'(\beta)S_2(\beta)$ or

$$C(\beta) = \sum_{j=1}^{n} \frac{t_j\beta}{1+t_j\beta} - \sum_{j=1}^{n} \ln(1+t_j\beta) \sum_{i=1}^{k} \frac{1}{1+t_i\beta}$$

where $t_1,\ldots,t_k$ are failure times and $t_{k+1},\ldots,t_n$ are censored life times. We introduce notation for the sample moments as follows:

$$\eta_r = \frac{1}{k}\sum_{i=1}^{k} t_i^{\,r}, \quad \zeta_r = \frac{1}{n}\sum_{j=1}^{n} t_j^{\,r} \quad \text{for } r = 1,2,3,\ldots, \tag{17}$$

then using the two expansions, valid for $|x| < 1$,

$$\ell n(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots, \quad \frac{1}{1+x} = 1 - x + x^2 - \ldots$$

and substituting into C and simplifying we find, upon neglecting terms of third order in β, that

$$(1+\eta_1\beta+\eta_2\beta)(\zeta_1+\zeta_2\frac{\beta}{2}+\zeta_3\frac{\beta^2}{3}) - [\zeta_1-\zeta_2\beta+\zeta_3\beta^2] = 0$$

Multiplying the first two together and collecting terms yields

$$(\frac{\zeta_2}{2} - \eta_1\zeta_1)\beta - (\zeta_3 - \eta_2\zeta_1 - \frac{\eta_1\zeta_2}{2})\beta^2 = 0$$

Now we notice that the condition equation (15), can be written in the notation of (17) as $\zeta_2 > 2\eta_1\zeta_1$.

Thus our computational procedure to decide upon the parametric representation of the distribution governing the observations which have been obtained is contained in the following.

Algorithm: Given $t_1,\ldots,t_k$ as failure times and $t_{k+1},\ldots,t_n$ as censored times from a $J_1(\alpha,\beta)$ distribution

(i) Compute the sample moments η_1, η_2, ζ_1, ζ_2, ζ_3.

(ii) If $\zeta_2 < 2\eta_1\zeta_1$, assume observations are from $J_2(\gamma,0)$ i.e. constant failure rate, distribution and estimate γ by

$$\hat{\gamma} = \frac{k}{n\zeta_1} .$$

(iii) If $\zeta_2 > 2\eta_1\zeta_1$, assume observations are from $J_1(\alpha,\beta)$ distribution and compute

$$\beta_0 = \frac{\zeta_2 - 2\eta_1\zeta_1}{2\zeta_3 - 2\eta_2\zeta_1 - \eta_1\zeta_2}$$

then use the Newton-Raphson interation procedure, namely for n = 0, 1, 2, ...

$$\beta_{n+1} = \beta_n - \frac{C(\beta_n)}{C'(\beta_n)}, \quad \hat{\beta} = \lim_{n \to \infty} \beta_n , \text{ and}$$

$$\hat{\alpha} = \frac{k}{\sum_{j=1}^{n} \ell n(1+t_j\hat{\beta})} .$$

Practical experience indicates that the iteration converges very rapidly. Since the functions are very simple a small programmable electronic calculator, such as the HP-65, can be used to obtain these estimates. Programs for the HP-65 and HP-97 are available from the authors.

7. Conclusion

If a component has a life distribution with an increasing failure rate, the information necessary to estimate its parameters must contain failure times. In practice this means that virtually no observed failures, within a fleet of operational components, provide little information with which to assess reliability.

If a component has a constant failure rate then both failure times and alive times contribute equally to its estimation. The preceeding study suggests that if a component has a life distribution with decreasing failure rate it is the alive times within the data which contribute principally to the estimation of the parameters.

The problem of obtaining the usual sampling distributions of the maximum likelihood estimators of the parameters for the decreasing failure rate model studied seems to be difficult because the estimates are only implicitly defined. Moreover, the usual proofs for the asymptotic normality of the MLE's, which define the asymptotic mean and variance, do not apply even when censoring is type I or type II. A useful asymptotic theory must be developed for the general censoring model.

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ATTACHMENT C

COMPARISON OF PARAMETER ESTIMATION

FOR THE PARETO DISTRIBUITON *

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ABSTRACT

The maximum likelihood estimate (MLE) for the shape parameter of the mixed exponential (Pareto, second kind) distribution is compared to the best linear unbaised estimate (BLUE), asymptotically best linear unbiased estimate (ABLUE) and the best linear unbiased estimate based on k order statistics ($BLUE_k$). The MLE is seen to be more flexible and at least as accurate as the BLUE. For censored data the MLE is generally more accurate than the ABLUE or $BLUE_k$. Estimation of the distribution function using joint MLE's of the scale and shape parameters compared favorably with the estimation of the distribution function using $BLUE_k$ for the scale parameter and known shape parameter.

Key words:

Pareto distribution
Mixed Exponential distribution
Simulation
Best linear unbiased estimation
Maximum likelihood estimation
Censored data

COMPARISON OF PARAMETER ESTIMATION

FOR THE PARETO DISTRIBUTION

1. INTRODUCTION

The mixed exponential distribution, often called the Pareto or Lomax distribution, has the distribution function:

$$F(x) = 1 - (1+\beta x)^{-\alpha}$$

This distribution has proven to be useful in reliability applications for representing the distribution of failure times of electronic equipment (Myhre and Saunders, 1976).

The estimation of the parameters of this distribution has been discussed in numerous papers. Kulldorff and Vännman (1973) have derived the best linear unbiased estimate (BLUE) of the scale parameter, $1/\beta$, for known values of the shape parameter, $\alpha > 2$. For known $\alpha \leq 2$ an asymptotic best linear unbiased estimator (ABLUE) for $1/\beta$ is also derived. These procedures require a complete sample; however, a censored sample may be used by ignoring the times after truncation. Vännman (1976) more recently has derived for known alpha a BLUE estimate of $1/\beta$ for censored data.

The only restriction of the procedure is that $k < n + 1 - 2/\alpha$ where n is the sample size and k is the number of failures. Each of these estimation procedures assumes the knowledge of the shape parameter. BLUE estimates of the shape parameter are not attainable.

Harris and Singpurwalla (1968) used the method of moments as an estimation procedure but this again was for a complete sample and with the restriction of $\alpha > 2$. In both Harris and Singpurwalla (1968) and Kulldorff and Vännman (1973) it is claimed that maximum likelihood estimation is difficult to obtain. However, in a later paper Harris and Singpurwalla (1969) exhibit the maximum likelihood equations for complete samples.

In a paper by Myhre and Saunders (1976) maximum likelihood estimates are derived for both the shape parameter, α, and the scale parameter, β, for censored or complete data. The joint MLE of α and β for a sample consisting of k failure times $t_1, \ldots t_k$, and n-k success, times $t_{k+1} \ldots t_n$, requires the numeric solution of:

$$H(\beta) = \frac{1}{k} \sum_{i=1}^{k} (1+t_i\beta)^{-1} \sum_{j=1}^{n} \ln(1+t_j\beta) - \sum_{j=1}^{n} \frac{t_j\beta}{1+t_j\beta} = 0$$

to obtain $\hat{\beta}$ while $\hat{\alpha} = \dfrac{k}{\sum_{i=1}^{n} \ln(1+t_i\beta)}$.

A sufficient condition for $H(\beta) = 0$ for some $\beta \neq 0$ is that

$$\frac{2}{k}\sum_{i=1}^{k} t_i \sum_{j=1}^{n} t_j < \sum_{j=1}^{n} t_j^2 .$$

The MLE of β given α for such a sample requires the solution of:

$$A(\beta) = \alpha \sum_{j=1}^{n} (1+t_j\beta)^{-1} + \sum_{i=1}^{k} (1+t_i\beta)^{-1} - n\alpha = 0 .$$

As $A(0) = k$ and as $A(\beta)$ is a decreasing function with $\lim_{\beta\to\infty} A(\beta) = -n\alpha$, a unique solution, $\hat{\beta}$, always exists. While the MLE's are not in general suitable for hand calculation, their existence is not restricted by the type of censoring or by the requirement that alpha be known.

In this paper the MLE procedure of Myhre and Saunders (1976) is compared with the BLUE and ABLUE procedures described in Kulldorff and Vännman (1973) and Vännman (1976). For complete samples with $\alpha > 2$ we know (Kuldorff and Vännman, 1973) that the BLUE and the MLE have an asymptotic relative efficiency of one. However, for reliability applications it is usually found that $\alpha \leq 2$ and that the sample is

censored. For these cases the variance of the $BLUE_k$ does not exist and hence no relative efficiency with the MLE can be computed. Therefore, comparisons are made using simulation techniques with particular attention to censored data and $\alpha \leq 2$.

2. SIMULATION MODEL

In a simulation comparing the BLUE (Kulldorff and Vännman, 1973) and the MLE of $1/\beta$, n failure times are generated from the distribution

$G(x) = 1 = (1+x\beta)^{-\alpha}$ for fixed α and β.

The times are ordered and for complete samples the two estimates are computed. In the case of data censored on the k^{th} ordered observation (Type II censoring), the MLE uses all the data whereas the BLUE used only the first k failure times (observations). For complete samples, $\alpha > 2$, the BLUE was no better an estimate than the MLE and for censored data the MLE was always a closer extimate than the BLUE. No other simulations using the BLUE procedure (Kulldorff and Vännman, 1973) were made.

In simulations comparing the ABLUE of Kulldorff and Vännman (1973) and the MLE of Myhre and Saunders (1976) for Type II censoring, n observations are generated from a fixed α, β, ordered and the first ten are

kept while the rest are set equal to the tenth time. The ABLUE estimates of $1/\beta$ are then made and compared with the MLE. The truncation is on the tenth ordered observation as ten is the largest number for which computational tables are available in Kulldorff and Vännman (1973). These simulations are run for $n = 50$, 100, 200; $\alpha = .5, 1.0, 1.5, 2.0, 2.5$; and selected values of β. The ABLUE was a more difficult procedure to use than the BLUE, but the MLE was again the superior estimator.

The BLUE developed in Vännman (1976), to be referred to as the BLUE_k, is a more versatile estimate than the BLUE and ABLUE developed in Kulldorff and Vännman (1973) and was therefore the subject of a more thorough investigation. In order to reflect practical applications in which tests are not only truncated on time (Type I censoring) or number of failures (Type II censoring) but are also randomly truncated, simulations were conducted for all these types of censoring. The BLUE_k is based on k order statistics, where k is known. It is possible, however, to compute an estimate of $1/\beta$ based on the BLUE_k procedure using Type I or using random censored data by letting k be the number of failures (observations). While this procedure is not mathematically precise, it will be shown that the

relative accuracy between the $BLUE_k$ and MLE for random censoring is not much less than the relative accuracy for Type II censoring.

The distribution function

$$Q(\tau) = \begin{cases} \dfrac{(\tau+D-T)^2}{(m+1)D^2} & T-D \le \tau \le T \\ 1-\dfrac{(T+mD-\tau)^2}{m(m+1)D^2} & T < \tau \le T+mD \end{cases},$$

for positive m, D and T, simulates random truncation times where $m > 1$ makes the distribution skewed to the right and $m < 1$ makes it skewed to the left. The spread of the truncated times is given by $(m+1)D$. Using this distribution for a fixed T, D, and m, a truncation time τ_i is generated for each item on test. If the generated failure (observation) time, t_i, is greater than τ_i, then the item is a success and t_i is set to τ_i, while if $t_i \le \tau_i$ it is counted as a failure and is unchanged. If $D = 0$ this is truncation on a fixed time $\tau = T$ (Type I censoring).

The simulation comparing the $BLUE_k$ and the MLE of $1/\beta$ was run with a variety of α's and β's using Type I, Type II and random censoring. For each censoring type and (α,β) pair at least twenty-eight simulations were run. From these simulations the MLE and $BLUE_k$ were

calculated and a determination made as to which estimate was closer to the true value of $1/\beta$. For each set of twenty-eight or more simulations the percentage of time the MLE is closer to $1/\beta$ is calculated and entered in Table 1.

Insert Table 1 about here

As shown, only values for α and β between zero and two were used. These values were chosen because reliability tests with real data showed that in general α is less than two and the $BLUE_k$ appears to be less accurate for large values of α. Also, for fixed α the relative accuracy of the $BLUE_k$ and MLE does not appear to change much as a function of β. Tests generally indicated that the sample size had no effect on the relative accuracy of the different estimation procedures, so a sample size of 100 was used for all simulations. Note, however, that the number of failures per simulation observed varied from 1 to over 50. For Type I censoring (Type II censoring) three different truncation times (failures) were used in order to check the relative accuracy of the MLE and $BLUE_k$ as a function of truncation time (failures). In both cases it was found that the relative accuracy was not

dependent on the truncation time (failures). For random censoring the relative accuracy of the MLE increases as the random censoring becomes less like Type I.

In a final type of simulation data is generated for a fixed α,β with Type I, Type II, and random censoring. The BLUE_k of $1/\beta$ for <u>given</u> α and the <u>joint</u> MLE of α and $1/\beta$ are computed. For these simulations the estimated distributions were compared with the true. This was done because the estimated α,β could be inaccurate while the estimated distribution still tracks well. Each estimated distribution, $F_M(x)$ and $F_B(x)$, is compared to the true, $F(x)$, at ten values from the interval $(F^{-1}(0), F^{-1}(.2)]$.

Define:

$$S_M = \sum_{i=1}^{10} |F_M(x_i)-F(x_i)|$$

$$S_B = \sum_{i=1}^{10} |F_B(x_i)-F(x_i)|$$

A determination is then made as to which estimated distribution is closer to the true according to which value, S_M or S_B, is smaller. For each set of thirty or more simulations, the percentage of the time the MLE is

closer to the true distribution is calculated and entered in Table 2.

Insert Table 2 about here

From Table 2 it is seen that for the cases examined the accuracy of the maximum likelihood estimate of the Pareto distribution function using joint estimation for α and β compares surprisingly well with the accuracy of the estimate obtained by assuming that α is known and using the $BLUE_k$ estimate for $1/\beta$.

3. CONCLUSION

The comparisons discussed in this paper are based on over 44,500 simulations, where each simulation used a sample of size 100. The number of failures per sample varied from 1 to 100 depending on the type of censoring being used.

For complete samples, with known $\alpha > 2$, the BLUE (Kulldorff and Vännman, 1973) of the scale parameter, $1/\beta$, was no better than the MLE. For censored samples, whether Type II, Type I or random, simulations show that for known $\alpha \leq 2$, the MLE is generally a more accurate estimate of the scale parameter than is the $BLUE_k$ (Vännman, 1976). As the shape parameter, α, increases the relative accuracy of the MLE increases.

However, in practice the shape parameter, α, is not known and the parameters α and β must be estimated jointly. This joint estimation (or even estimation of α with known β) is not possible with BLUE, ABLUE, or $BLUE_k$ methods. Therefore, in practice the joint MLE procedure would be used even if the MLE of $1/\beta$ was not more accurate than the $BLUE_k$ (BLUE). However, the MLE for $1/\beta$ is generally more accurate than the $BLUE_k$ and in addition the estimation of the distribution function using joint MLE's of the scale and shape parameters compare favorably with the estimation of the distribution function using the $BLUE_k$ for the scale parameter and <u>known</u> shape parameter.

1. Comparison of $BLUE_k$ with MLE for $1/\beta$ (α Given): Percentage of Times MLE is Closer to $1/\beta$

β	α = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.2	1.4	1.6	1.8	2.0
a. Type I Censoring — 30 simulations, n = 100; 10 simulations with $\tau = F^{-1}(0.1)$; 10 simulations with $\tau = F^{-1}(0.2)$; 10 simulations with $\tau = F^{-1}(0.5)$															
0.1	56	50	63	56	50	73	90	80	70	86	73	93	93	83	90
0.2	60	66	43	60	83	60	70	73	73	80	90	86	93	86	93
0.3	60	53	70	73	56	73	76	70	73	76	80	76	83	96	100
0.4	56	56	73	66	56	70	70	53	73	80	83	90	86	83	100
0.5	43	73	66	70	56	60	73	76	80	80	73	93	86	86	96
0.6	60	73	46	60	56	70	66	46	80	73	80	90	93	73	96
0.7	56	60	43	66	53	70	63	70	73	76	90	76	83	90	93
0.8	53	46	50	56	63	83	73	73	80	83	76	90	76	96	86
0.9	53	50	60	50	73	66	70	76	83	83	86	93	83	90	83
1.0	63	50	53	56	53	63	73	86	93	76	90	83	86	86	96
1.2	66	66	50	63	76	56	73	76	76	70	83	83	86	93	93
1.4	60	50	46	73	63	60	76	73	76	76	80	86	93	100	86
1.6	46	53	50	70	80	76	70	83	80	80	76	86	90	83	100
1.8	66	56	50	60	70	73	63	63	60	86	76	80	90	96	96
2.0	53	53	43	73	66	70	76	76	83	83	80	86	96	90	96

Table 1. (continued)

b. Type II Censoring

28 simulations n = 100

9 simulations with k = 10

9 simulations with k = 20

10 simulations with k = 50

0.1	46	53	53	64	64	67	75	78	75	67	82	85	82	92	89
0.2	50	46	57	50	53	64	71	71	89	75	78	85	92	89	89
0.3	57	46	46	53	67	60	78	60	82	71	85	82	82	92	96
0.4	53	60	50	60	75	78	64	75	89	75	71	96	82	96	82
0.5	53	46	42	53	57	57	67	78	67	89	89	85	92	96	78
0.6	67	78	60	60	57	71	75	57	71	75	85	96	92	100	85
0.7	46	28	42	57	64	71	64	82	89	82	89	92	92	71	89
0.8	50	46	57	35	64	57	46	75	64	89	82	96	96	78	85
0.9	53	64	57	64	50	53	60	67	64	82	89	89	96	92	82
1.0	60	42	53	67	75	60	60	71	82	75	92	82	92	89	89
1.2	50	46	35	67	75	67	85	64	85	82	89	78	89	96	92
1.4	50	57	57	60	50	71	64	71	75	78	92	85	92	89	89
1.6	60	60	50	39	53	71	85	64	78	78	85	85	89	96	85
1.8	60	46	78	57	42	53	85	75	85	85	82	82	89	85	85
2.0	75	42	57	67	57	53	60	78	67	71	78	78	92	82	100

Table 1. (continued)

c. Random Censoring

40 simulations $n = 100$ $m = 3$

10 simulations with $D = 0.1$, $T = F^{-1}(0.1)$

10 simulations with $D = 0.1$, $T = F^{-1}(0.2)$

10 simulations with $D = 0.2$, $T = F^{-1}(0.1)$

10 simulations with $D = 0.2$, $T - F^{-1}(0.2)$

0.1	60	60	52	67	72	55	75	75	80	82	82	95	95	90	100
0.2	57	52	57	57	65	75	82	62	80	92	85	92	92	92	87
0.3	82	42	57	55	62	77	77	75	72	75	85	82	97	92	92
0.4	47	52	45	70	67	72	62	82	77	82	82	92	80	97	100
0.5	52	65	65	57	65	77	67	77	82	85	87	92	95	92	95
0.6	55	55	40	52	62	57	87	72	80	82	87	87	92	87	90
0.7	55	52	50	62	67	70	62	80	82	82	90	87	87	97	92
0.8	55	62	37	77	70	70	75	75	70	85	95	85	97	95	92
0.9	52	65	70	67	70	77	85	87	77	80	77	95	100	97	92
1.0	47	47	52	67	65	70	75	85	72	70	77	92	87	95	92
1.2	52	55	52	57	65	72	75	85	80	95	85	92	95	90	92
1.4	60	45	42	62	52	65	70	87	80	67	82	87	92	92	92
1.6	55	57	60	62	70	70	72	82	87	85	90	87	95	87	97
1.8	52	55	52	57	67	75	75	75	92	65	82	87	90	87	92
2.0	52	70	57	55	67	82	70	70	80	85	77	82	85	90	90

2. Comparison of $BLUE_k$ for $1/\beta$ (α Given) with Joint MLE of α and $1/\beta$: Percentage of Times Distribution Estimated with MLE's is Closer to the True Distribution

β	α 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.2	1.4	1.6	1.8	2.0
	a. Type I Censoring 30 simulations n = 100 10 simulations with $\tau = F^{-1}(0.1)$ 10 simulations with $\tau = F^{-1}(0.2)$ 10 simulations with $\tau = F^{-1}(0.5)$														
0.1	37	57	37	23	27	43	27	50	57	73	47	47	47	43	60
0.2	57	30	40	33	40	43	47	43	47	43	37	63	57	60	57
0.3	37	43	37	40	37	43	50	57	63	53	57	43	47	70	60
0.4	50	40	50	53	40	33	23	37	33	47	50	50	50	60	53
0.5	30	50	27	23	47	43	50	53	53	50	60	57	67	50	47
0.6	37	47	53	23	20	43	53	40	50	70	60	60	60	67	63
0.7	43	43	53	30	37	33	47	50	50	53	53	70	47	50	57
0.8	47	43	37	30	40	40	53	43	43	47	73	63	67	57	53
0.9	53	33	40	50	37	63	43	63	57	53	63	77	63	70	60
1.0	37	30	40	33	43	53	30	50	50	63	53	60	50	47	57
1.2	33	47	43	33	37	50	57	30	43	47	57	57	63	47	53
1.4	40	37	30	37	37	40	40	33	53	57	50	57	50	60	67
1.6	43	57	37	33	53	57	40	50	50	60	67	67	67	80	47
1.8	53	47	43	40	40	40	53	57	67	40	53	57	40	63	60
2.0	37	50	30	23	40	53	60	50	47	57	50	63	67	53	53

Table 2. (continued)

b. Type II Censoring

30 simulations n = 100

10 simulations with k = 10

10 simulations with k = 20

10 simulations with k = 50

0.1	30	53	33	30	30	23	43	40	40	47	40	43	50	50	40
0.2	30	37	20	37	33	33	33	50	53	27	53	33	53	43	50
0.3	33	40	37	33	30	27	40	60	33	37	50	50	53	33	40
0.4	33	17	33	37	40	37	47	43	47	43	37	37	53	27	37
0.5	53	30	30	37	30	43	33	50	57	30	47	40	43	40	47
0.6	47	27	23	40	37	43	60	50	30	43	53	37	43	47	40
0.7	23	43	27	33	43	37	33	37	57	47	53	37	47	47	37
0.8	30	37	30	23	27	50	37	43	33	43	37	50	40	50	37
0.9	30	27	37	23	30	37	27	40	33	43	53	47	40	50	23
1.0	23	33	33	33	30	43	30	47	57	33	50	50	50	23	43
1.2	40	47	37	40	37	10	43	43	40	40	50	57	43	53	43
1.4	37	33	40	23	30	43	27	53	53	53	37	43	63	50	30
1.6	33	30	33	40	23	47	47	50	27	53	40	37	37	23	53
1.8	53	33	37	37	37	20	33	43	63	47	43	50	37	40	47
2.0	33	23	30	53	17	20	23	27	37	37	40	57	40	50	37

Table 2. (Continued)

c. Random Censoring

40 simulations n = 100 m = 3

10 simulations with D = 0.1, $T = F^{-1}(0.1)$

10 simulations with D = 0.1, $T = F^{-1}(0.2)$

10 simulations with D = 0.2, $T = F^{-1}(0.1)$

10 simulations with D = 0.2, $T = F^{-1}(0.2)$

0.1	35	35	33	35	25	33	35	38	40	50	53	53	45	35	53
0.2	48	40	30	30	43	58	35	43	48	38	63	68	48	48	55
0.3	30	40	40	40	38	35	43	55	45	40	45	45	45	50	60
0.4	35	40	25	45	35	38	48	43	58	45	53	60	60	45	45
0.5	33	35	53	35	53	40	30	60	45	43	50	40	58	38	55
0.6	38	43	28	38	48	43	53	40	40	35	60	53	58	63	40
0.7	25	38	35	30	28	40	45	48	53	60	43	60	48	50	60
0.8	48	25	28	23	33	38	53	33	33	40	55	55	45	53	48
0.9	33	45	33	30	48	48	53	55	63	53	58	45	38	38	55
1.0	43	28	35	25	55	43	35	38	45	45	43	28	40	58	50
1.2	38	25	25	30	40	38	53	45	53	43	50	48	40	58	58
1.4	33	38	38	43	40	38	63	43	53	63	58	50	60	70	50
1.6	35	48	28	33	30	43	33	48	48	45	45	45	45	55	50
1.8	48	28	33	33	28	28	55	35	43	55	43	63	45	38	48
2.0	33	45	25	38	23	35	33	35	48	50	55	45	30	60	53

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ATTACHMENT D

HP-97 PROGRAMS FOR THE COMPUTATION OF MAXIMUM LIKELIHOOD ESTIMATES FOR THE PARAMETERS OF THE MIXED EXPONENTIAL DISTRIBUTION

by

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HP-97 PROGRAMS FOR THE COMPUTATION OF MAXIMUM LIKELIHOOD ESTIMATES FOR THE PARAMETERS OF THE MIXED EXPONENTIAL DISTRIBUTION

Reliability models for electronic components must be flexible enough to handle both complete and truncated data and the computations involved should be straight-forward. For these reasons, as well as its appealing intuitive properties, the exponential model has been used extensively. However, in many applications it cannot be assumed that the components have a constant failure rate and for these cases the mixed exponential model has been developed. This model can be thought of resulting from a mixture of constant failure rates and results in the distribution $F(x) = 1 - (1 + \beta x)^{-\alpha}$ with reliability $R(t) = (1 + \beta t)^{-\alpha}$ and a decreasing failure rate. This model has been studied in reports 1, 2, 3, 4; and in 2 it is seen that the maximum likelihood estimates of α and β are preferable.

The M.L.E. of β and α for a sample of k failures $t_1, t_2, \ldots, t_k$ and n=k alive times, $t_{K+1}, \ldots, t_n$ requires the numeric solution of

$$H(\beta) = 1/k \sum_{i=1}^{k} (1 + t_i\beta)^{-1} \sum_{j=1}^{n} \ln(1 + t_j\beta) - \sum_{j=1}^{n} \frac{t_j\beta}{1+t_j\beta} = 0$$

for $\hat{\beta}$ and $\hat{\alpha} = \dfrac{k}{\sum_{i=1}^{n} \ln(1+t_i\hat{\beta})}$.

A sufficient condition for $H(\beta) = 0$ for some $\beta > 0$ is that $2/k \left(\sum_{i=1}^{k} t_i\right)\left(\sum_{j=1}^{n} t_j\right) < \sum_{j=1}^{n} t_j^2$. If $\hat{\beta}$ cannot be found then the exponential model should be used with $\hat{\lambda} = k/\left(\sum_{1}^{n} t_i\right)$.

As the solution of $H(\beta) = 0$ can only be attained by iterative methods, programs for the HP-97 have been developed which allow the user to compute $\hat{\beta}$ and $\hat{\alpha}$ for a test of not over twenty items. The first program, A, will test the sufficient conditions for the solution. The second program, B, will use interval halving to approximate $\hat{\beta}$, the solution of $H(\beta) = 0$, accurate to within an error of .0001. The third program, C, will use this $\hat{\beta}$ to approximate $\hat{\alpha}$.

These programs will allow the mixed exponential model to be used as extensively as the exponential model in applications of reliability theory.

INSTRUCTIONS

Store n, the number of items on test in E, and k the number of failures in D. Store the time t_i, failures first, in the registers one through n, where $n \leq 20$. After the data has been input the programs may be run in order without re-entry of the times.

Program A will return + 1 if the sufficient conditions are met and the user should run program B. If the conditions are not met, -1 will be returned and the reliability is estimated as $R(t) = e^{-\hat{\lambda} t}$ where $\hat{\lambda} = \frac{k}{\sum_{i=1}^{n} t_i}$.

Program B will return an approximation of $\hat{\beta}$ with error $\leq$ .0001. If no estimate occurs between 0 and 50 the program returns a negative and, as above, the reliability should be computed as $R(t) = e^{-\hat{\lambda} t}$.

Program C will return $\hat{\alpha}$ based on the data and the $\hat{\beta}$ found by program B.

PROGRAM A

001	*LBLA	21 11
002	RCLE	36 15
003	X⇄I	16-41
004	*LBL7	21 07
005	RCLi	36 45
006	+	-55
007	DSZI	16 25 46
008	GTO7	22 07
009	RCLD	36 14
010	X⇄I	16-41
011	*LBL8	21 08
012	RCLi	36 45
013	+	-55
014	DSZI	16 25 46
015	GTO8	22 08
016	×	-35
017	2	02
018	×	-35
019	RCLD	36 14
020	÷	-24
021	RCLE	36 15
022	X⇄I	16-41
023	*LBL9	21 09
024	RCLi	36 45
025	X²	53
026	+	-55
027	DSZI	16 25 46
028	GTO9	22 09
029	-	-45
030	X<0?	16-45
031	GTOE	22 15
032	1	01
033	CHS	-22
034	PRTX	-14
035	RTN	24
036	*LBLE	21 15
037	1	01
038	PRTX	-14
039	RTN	24

PROGRAM B

040	R/S	51
041	*LBLB	21 12
042	5	05
043	0	00
044	STOA	35 11
045	GSBd	23 16 14
046	STOC	35 13
047	RCLA	36 11
048	STOB	35 12
049	EEX	-23
050	CHS	-22
051	6	06
052	STOA	35 11
053	GSBd	23 16 14
054	RCLC	36 13
055	×	-35
056	X<0?	16-45
057	GTOc	22 16 13
058	CHS	-22
059	RTN	24
060	*LBLc	21 16 13
061	RCLA	36 11
062	RCLB	36 12
063	+	-55
064	2	02
065	÷	-24
066	STOA	35 11
067	GSBd	23 16 14
068	RCLC	36 13
069	×	-35
070	X<0?	16-45
071	GTO5	22 05
072	RCLC	36 13
073	÷	-24
074	STOC	35 13
075	RCLB	36 12
076	RCLA	36 11
077	STOB	35 12
078	2	02
079	×	-35
080	X⇄Y	-41
081	-	-45
082	STOA	35 11
083	*LBL5	21 05
084	RCLB	36 12
085	RCLA	36 11
086	-	-45
087	EEX	-23
088	CHS	-22
089	4	04
090	-	-45
091	X>0?	16-44
092	GTOc	22 16 13
093	RCLA	36 11
094	RCLB	36 12
095	+	-55
096	2	02
097	÷	-24
098	RTN	24
099	*LBLd	21 16 14
100	RCLE	36 15
101	X⇄I	16-41
102	*LBL1	21 01
103	RCLi	36 45
104	RCLA	36 11
105	×	-35
106	1	01
107	+	-55
108	LN	32
109	+	-55
110	DSZI	16 25 46
111	GTO1	22 01
112	RCLD	36 14
113	X⇄I	16-41
114	*LBL2	21 02
115	RCLi	36 45
116	RCLA	36 11
117	×	-35
118	1	01
119	+	-55
120	1	01
121	X⇄Y	-41
122	÷	-24
123	+	-55
124	DSZI	16 25 46
125	GTO2	22 02
126	×	-35
127	RCLA	36 11
128	÷	-24
129	RCLD	36 14
130	÷	-24
131	RCLE	36 15
132	÷	-24
133	RCLE	36 15
134	X⇄I	16-41
135	*LBL3	21 03
136	RCLi	36 45
137	RCLA	36 11
138	×	-35
139	1	01
140	+	-55
141	RCLi	36 45
142	X⇄Y	-41
143	÷	-24
144	+	-55
145	DSZI	16 25 46
146	GTO3	22 03
147	RCLE	36 15
148	÷	-24
149	-	-45
150	RTN	24

PROGRAM C

151	DSP6	-63 06
152	*LBLC	21 13
153	RCLE	36 15
154	X⇄I	16-41
155	*LBL4	21 04
156	RCLA	36 11
157	RCLi	36 45
158	×	-35
159	1	01
160	+	-55
161	LN	32
162	+	-55
163	DSZI	16 25 46
164	GTO4	22 04
165	RCLD	36 14
166	X⇄Y	-41
167	÷	-24
168	PRTX	-14
169	RTN	24
170	R/S	51

ATTACHMENT E

1977 ASQC TECHNICAL CONFERENCE TRANSACTIONS—PHILADELPHIA

RELIABILITY TUTORIAL

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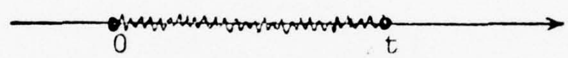
I. DISTRIBUTIONS IMPORTANT IN RELIABILITY THEORY

1. Basic Definitions

Let the random variables T denote the time to failure for a "new" component or structure.

The distribution function at time t is defined by

$$F(t) = P[T \le t]$$

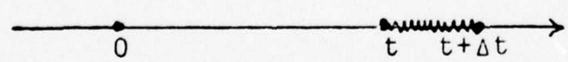


The reliability at time t is defined by

$$R(t) = P[T > t]$$

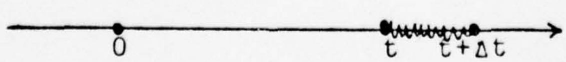
The conditional probability of failure during the interval $(t, t+\Delta t)$ is given by

$$F(\Delta t \mid t) = \frac{P[t < T \le t+\Delta t]}{P[T > t]}$$



Failure rate* at time t is defined by

$$r(t) = \lim_{\Delta t \to 0} \frac{F(\Delta t \mid t)}{\Delta t}$$



As $\Delta t \to 0$, $(t + \Delta t) \to t$.

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*r(t) is known by a variety of names such as hazard rate, force of mortality, intensity rate.

Research, in part, supported by the Office of Naval Research Contract N00014-76-C-0695.

If the probability density function, f(t),** exists, then

$$r(t) = \frac{f(t)}{R(t)}$$

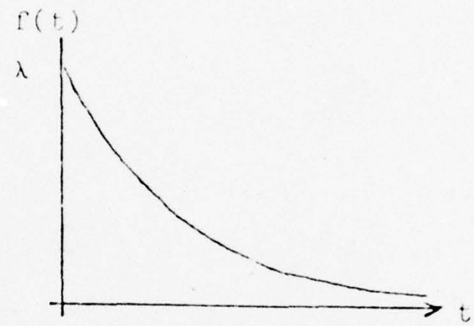
An increasing failure rate implies wear out. A decreasing failure rate is equivalent to a decreasing conditional probability of failure.

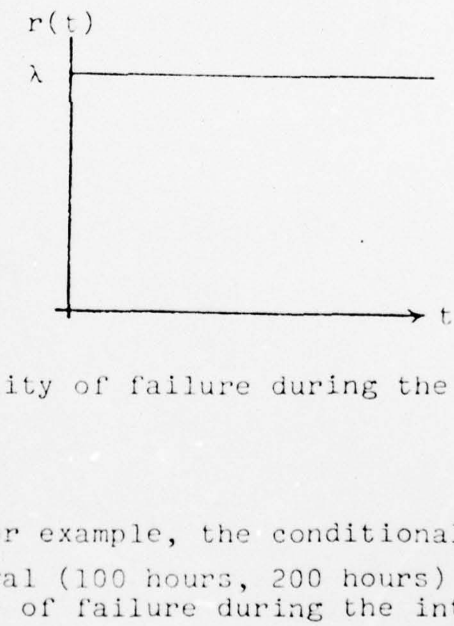
2. Distributions

2.1 Exponential Distribution

The probability density function is given by

$$f(t) = \lambda e^{-\lambda t} \qquad t \geq 0$$
$$= 0 \qquad \text{elsewhere}$$





It is easily shown that the failure rate $r(t) = \lambda$. The mean time to failure is $1/\lambda$.

It follows that the conditional probability of failure during the interval (t_o, t_o+t) is given by

$$F(t|t_o) = \lambda t$$

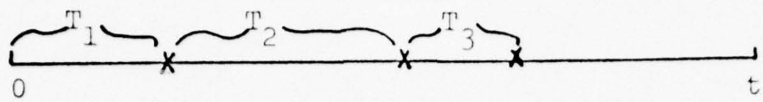
which is independent of the time t_o. For example, the conditional probability of failure during the interval (100 hours, 200 hours) is identical to the conditional probability of failure during the interval (1000 hours, 1100 hours).

2.2 Poisson Process

If the intervals between successive events are independent identically distributed according to an exponential distribution with failure rate λ then the number of events which occur during the interval [0,t] has a Poisson distribution with mean value λt. Also, if the num-

**Recall that $f(t) = \frac{dF(t)}{dt}$. Also $P(a < T < b) = \int_a^b f(t)dt$.

ber of events which occur during the interval $[0,t]$ has a Poisson distribution with mean value λt then the T_i are independent identically



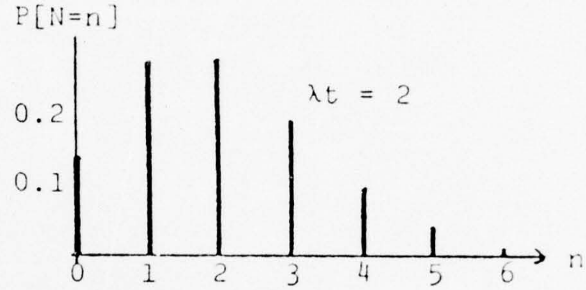
distributed as exponential variables with failure rate λ.

For example, if the intervals between successive events are independent identically distributed according to an exponential distribution with $\lambda = 4$/hour then the number of events in $[0,2]$ has a Poisson distribution with expected number of arrivals equal to (4/hour)(2 hours) = 8.

If N has a Poisson distribution then

$$P[N=n] = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

for $n = 0,1,2, \ldots$



If $\lambda = 4$ and $t = 2$ then

$$P[N=5] = \frac{(8)^5 e^{-8}}{5!} = .092.$$

2.2.1 Example: Maintained Units

Consider a battery which is put into operation at time 0. Each time a failure occurs the failed battery is replaced and a new battery is installed. If the life lengths of the batteries are independent, exponentially distributed with failure rate λ then the number of failures in $[0,t]$ has a Poisson distribution with expected value λt.

If $\lambda = 4$/year then the expected number of failures in one year is 4. The probability that the number of failures, N, in one year will be less than or equal to 4 is

$$P[N \le 4] = P[N=0] + P[N=1] + \ldots + P[N=4] = .639.$$

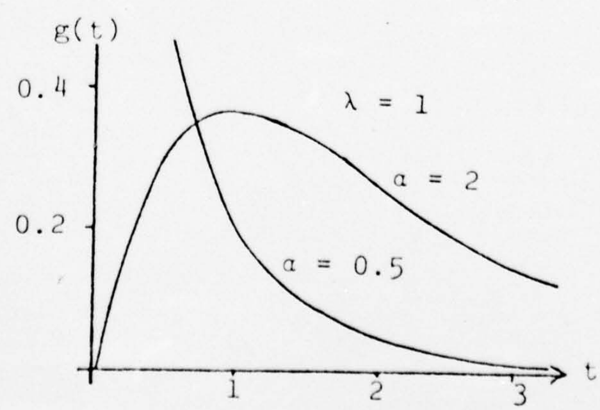
2.3 Gamma Distribution

The gamma distribution has probability density function

$$g(t) = \frac{\lambda^\alpha t^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)} \quad \text{for } t \ge 0$$

$$= 0 \qquad \text{elsewhere}$$

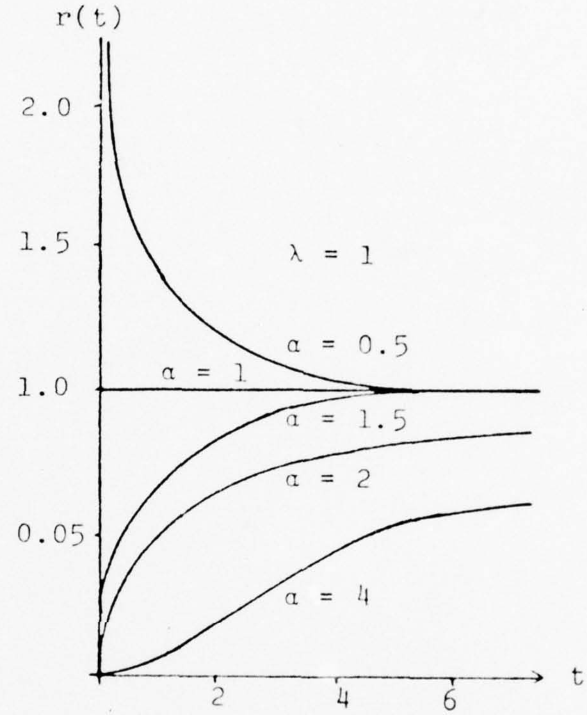
where $\lambda > 0$, $\alpha > 0$. α is the shape parameter. As α increases, the density function, g, becomes less peaked.



When α is an integer the gamma distribution may be thought of as the sum of α independent exponential random variables each with failure rate λ.

Assume that a device subject to an environment will fail when exactly $k \geq 1$ shocks occur and not before. If the shocks occur with a Poisson distribution with rate λ * then the waiting time until the item fails is described by a gamma distribution with parameters $\alpha = k$ and λ.

When $0 < \alpha < 1$, the failure rate is decreasing. When $\alpha > 1$, the failure rate is increasing. When $\alpha = 1$, the failure rate is constant, and the distribution is exponential. It can be shown that when $\alpha > 1$ the failure rate is bounded above by λ.



2.4 Weibull Distribution

The Weibull distribution is defined by

$$F(t) = 1-e^{-\lambda t^{\alpha}} \qquad t \geq 0$$

which is equivalent to

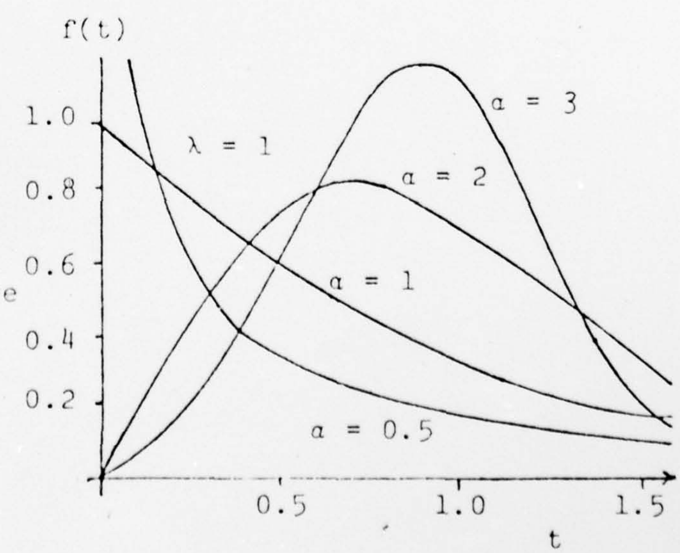
$$R(t) = e^{-\lambda t^{\alpha}}$$

and to

$$f(t) = \lambda \alpha t^{\alpha-1} e^{-\lambda t^{\alpha}} \qquad t \geq 0$$

$$= 0 \qquad \text{elsewhere}$$

where λ, $\alpha > 0$.

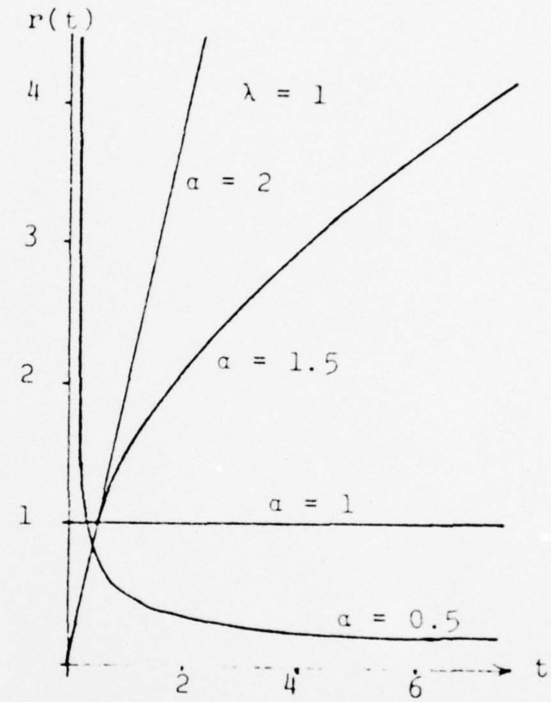


*This is equivalent to the assumption that the time between shocks has an exponential distribution with failure rate λ.

α is the shape parameter. As α increases $r(t)$ rises more steeply and the density function becomes more peaked. λ is the scale parameter. This is an extreme value distribution in that it approximates the limiting distribution of the minimum of independent random variables.

The Weibull distribution has been used to describe fatigue failure, vacuum tube failure, ball bearing failure, electronic equipment failure, etc.

When $0 < \alpha < 1$, the failure rate is decreasing. When $\alpha > 1$, the failure rate is increasing. When $\alpha = 1$, the failure rate is constant and the distribution is exponential. It can be shown that when $\alpha > 1$ the failure rate $r(t)$ is unbounded as t becomes infinite.



2.5 Mixed Exponential Distribution

Let X_λ be the life length of a component in a service environment with a constant failure rate λ which is unknown. The variability of manufacture determines various percentages of the λ values and this variability can be described by some distribution, say G.

Assume G is a gamma distribution. When λ is known the reliability is given by

$$R(t) = e^{-\lambda t} \qquad t \geq 0$$

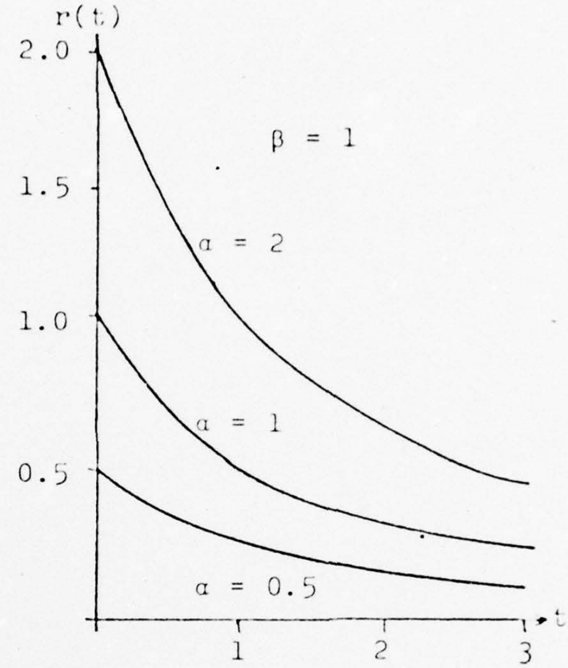
When λ is unknown

$$R(t) = \int_0^\infty e^{-\lambda t} g(\lambda)\, d\lambda = \frac{1}{(1+t\beta)^\alpha} = \exp\{-\alpha \ell n(1+t\beta)\}$$

The failure rate, $r(t)$, is given by

$$r(t) = \frac{\alpha\beta}{1+\beta t}$$

which decreases as t increases.



A burn in of h units of time on a component with an initial life which follows mixed exponential distribution with parameters α and β has a residual life which follows a mixed exponential distribution with parameters α and $\frac{\beta}{1+h}$. Note that for fixed t reliability increases as β (or α) decreases.

2.6 Bernoulli Distribution

Assume that the time to failure, T, for a new component follows some probability distribution F. If we fix time at t_o and are concerned only with the reliability at this time then reliability is no longer written explicitly as a function of time. If the distribution F is exponential then

$$R(t_o) = e^{-\lambda t_o} = p \ .$$

If the distribution F is Weibull then

$$R(t_o) = e^{-\lambda t_o^{\alpha}} = p \ .$$

Let $X = 1$ if $T > t_o$ and $X = 0$ if $T \le t_o$. The Bernoulli distribution has a density function given by

$$\begin{aligned} p(x) &= p & \text{if } x = 1 \\ &= 1 - p = q & \text{if } x = 0 \end{aligned}$$

The reliability is given by the value p.

II. POSSIBLE ERRORS MADE WHEN FAILURE RATES ARE DECREASING BUT ARE ASSUMED TO BE CONSTANT

If we assume a constant failure rate when in fact the failure rate is decreasing we tend to over-estimate the reliability for the time less than the average test time. Consider the following examples:

System Test for a Specific Type of Electronics Package
Average test time is 206 minutes.

time t minutes	mixed-exponential estimate of reliability at time t	constant failure rate estimate of reliability at time t
10	.99	1.00
100	.95	.97
200	.94	.95
300	.92	.92
1000	.87	.76

Note that for times much greater than the average test time the mixed exponential estimate of reliability is greater. However, estimates of reliability should not be made at times which are much greater than those for which data is available.

Environmental Test for a Specific Type of Electronics
Average test time is 234 minutes.

time t minutes	mixed-exponential estimate of reliability at time t	constant failure rate estimate of reliability at time t
10	.94	.98
100	.67	.80
300	.49	.50

Given data from two different lots we might incorrectly distinguish the more reliable lot if we assume constant failure rate. Consider the following example:

First Data Set

Failure times: 1, 8, 10
Alive times: 59, 72, 76, 113, 117, 124, 145, 149, 153, 182, 320.

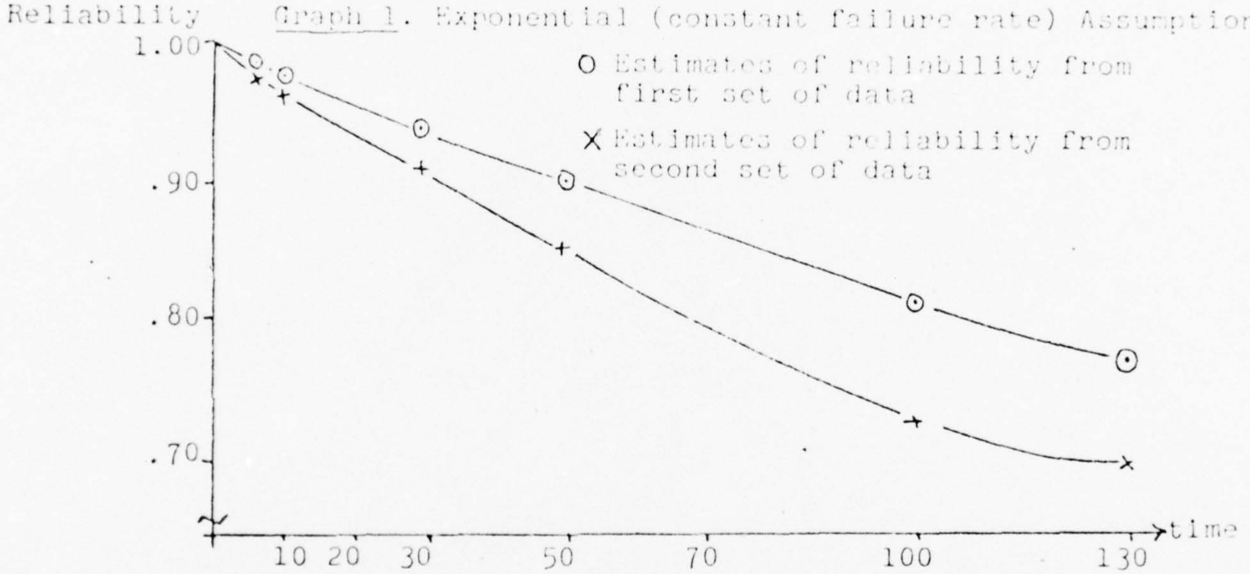
Second Data Set

Failure times: 37, 53
Alive times: 60, 64, 66, 70, 72, 96, 123.

If we assume that the data are observations from an exponential distribution (constant failure rate λ) then we have the following estimates:

time t in min.	Data Set 1 $\hat{\lambda}$ = .00197 estimate of reliability at time t, $\hat{R}_1(t)$	Data Set 2 $\hat{\lambda}$ = .00312 estimate of reliability at time t, $\hat{R}_2(t)$
6	.988	.981
10	.980	.969
30	.943	.911
50	.906	.856
100	.821	.732
130	.774	.667

Graph 1. Exponential (constant failure rate) Assumption

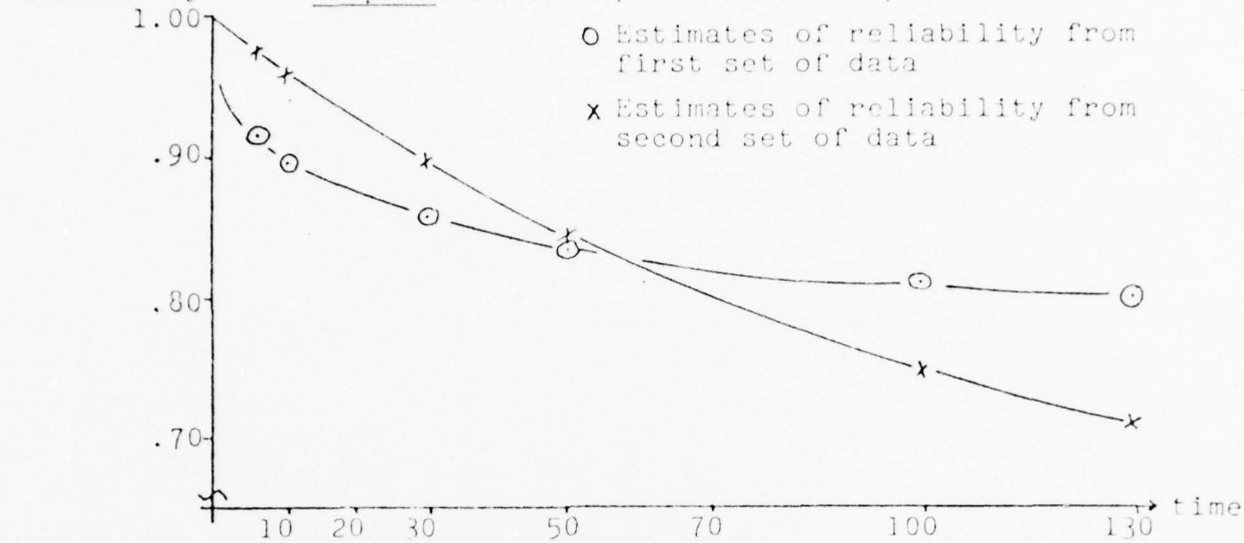


Looking at the data from the two sets we would expect that at least for the first fifty minutes the reliability estimate for the second set of data would be higher than the reliability estimate for the first set of data, because in the first set 3 failures out of 14 trials have occurred in the first ten minutes while in the second set only 1 failure out of 9 trials has occurred in the first fifty minutes. However, under the exponential assumption the reliability estimates for the first data set are consistently higher. See Graph 1.

If we assume that the data are observations from a mixed exponential distribution then we have the following estimates:

time t in min.	Data Set 1 estimate of reliability at time t, $\hat{R}_1(t)$	Data Set 2 estimate of reliability at time t, $\hat{R}_2(t)$
6	.915	.976
10	.896	.961
30	.855	.896
50	.836	.843
100	.810	.747
130	.801	.705

Graph 2. Mixed Exponential Assumption



Note that these estimates are more consistent with what the data show, that is, for at least the first 50 minutes we expect the reliability estimate for the second set of data to be higher than the reliability estimate for the first set of data. Beyond this time, however, say at 100 minutes, the data indicate that the reliability estimate from the first set of data should be higher than the reliability estimate from the second set of data. Using mixed exponential estimates this is the case. See Graph 2.

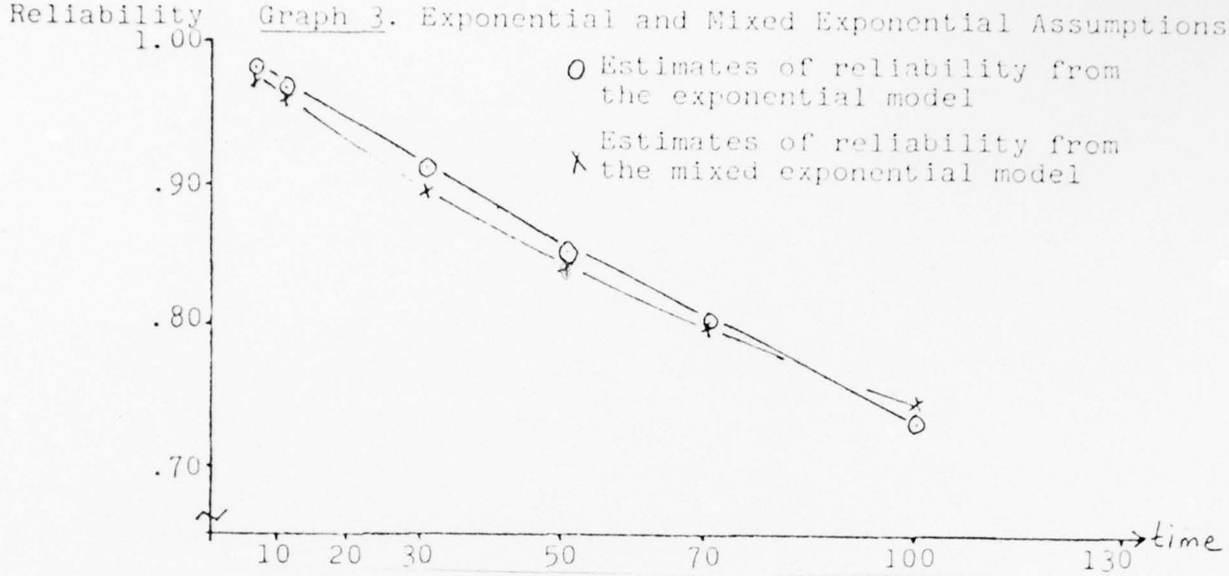
A statistical test to determine whether the data has a constant or decreasing failure rate was run on the data from sets 1 and 2. For data set 1 we reject constant failure rate in favor of decreasing failure rate. For data set 2 we cannot reject the constant failure rate assumption. In this case, however, the constant failure rate estimates for reliability and the mixed exponential estimates for reliability are close.

The mixed exponential estimates are conservative to about 70 minutes. (See Graph 3.) Actually one should not estimate reliability much beyond 70 minutes since we do not have data to support those estimates.

Data Set 2

time t in min.	exponential estimate of reliability at time t, R(t)	mixed exponential estimate of reliability R(t)
6	.981	.976
10	.969	.961
30	.911	.896
50	.856	.843
70	.804	.800
100	.732	.747

Graph 3. Exponential and Mixed Exponential Assumptions



III. GOODNESS OF FIT TESTS

1. General Goodness of Fit Tests

1.1 Kolmogorov-Smirnov Test Mann, N. R., Schafer, R. E., and Singpurwalla, N. D. Methods for Statistical Analysis of Reliability and Life Data, Wiley, 1974.

1) Must know parameters. That is, if we want to test whether or not the data is a sample from a Weibull distribution we must specify the values of the Weibull parameters, λ and α. For testing normality and for testing exponentiality this restriction of knowledge of the parameter values is not necessary if we use Lilliefor's alternative to Kolmogorov-Smirnov test.

2) In general the data must be uncensored.

3) Test is applicable to small samples.

1.2 Chi-square Goodness of Fit Tests Standard Statistical Text

1) Estimate the parameters from data.

2) Asymptotic method.

3) Data must be uncensored.

4) Requires a large sample size.

2. Tests for Constant Failure Rate Against Increasing or Decreasing Failure Rates

2.1 Finkelstein, J. M., and Schafer, R. E. (1971) Improved goodness-of-fit tests, Biometrika, 58, [illegible].

1) Uncensored data.

2) Good power.

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2.2 Gnedenko Test Gnedenko, B., et al., (1969) Mathematical Theory of Reliability, Academic Press.

1) Censored data.

2) Most powerful test.

2.3 Test of Fit for a Weibull Distribution Mann, et al., (1974)
Mann "LEAP" Test

1) Censored data.

2) Parameter values not specified.

3) Powerful test against checked alternatives.

IV. POINT ESTIMATES FOR COMPONENT RELIABILITY

1. Point Estimates for the Exponential Distribution

Example 1.1 Complete sample

Lex $X_1, \ldots, X_n$ be a random sample from a distribution with probability density function

$$f(x;\theta) = \lambda e^{-\lambda x} \qquad x > 0$$

The maximum likelihood estimate for λ is given by

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} X_i}$$

The maximum likelihood estimate for $R(t)$ is given by $e^{-\hat{\lambda} t}$.

Assume 4 items of a particular type are tested.

$x_1 = 30$ min.
$x_2 = 60$ min.
$x_3 = 75$ min.
$x_4 = 93$ min.

$$\hat{\lambda} = \frac{4}{\sum_{i=1}^{4} x_i} = \frac{4}{258} = .0155$$

$$\hat{R}(t) = e^{-\hat{\lambda} t} = e^{-.0155t}$$

If $t = 10$ minutes then $\hat{R}(10) = e^{-.155} = .86$

Example 1.2 Type II censoring (fixed number of failures)

Let $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ be an ordered sample from an exponential distribution with failure rate λ. Assume Type II censoring at the r^{th} failure.

Let the total tested life, T, be given by

$$T = \sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)} .$$

The maximum likelihood estimate for λ is

$$\hat{\lambda} = \frac{r}{T} .$$

The maximum likelihood estimate for R(t) is

$$e^{-\hat{\lambda} t} .$$

Assume ten items are put on test and tested until the 4th item fails.

$x_{(1)} = 30$ min, $x_{(2)} = 60$ min, $x_{(3)} = 75$ min, $x_{(4)} = 93$ min

Under the assumptions listed above the observed value of T is $30 + 60 + 75 + 93 + 6(93) = 816$.

$\hat{\lambda} = \frac{r}{T} = \frac{4}{816} = .005 \qquad \hat{R}(t) = e^{-\hat{\lambda} t} = e^{-.005t}$.

If t = 10 minutes then $\hat{R}(10) = e^{-.05t} = .952$.

Compare the results from Examples 1.1 and 1.2.

Example 1.1	Example 1.2
$x_1 = 30$ min.	$x_1 = 30$ min.
$x_2 = 60$ min.	$x_2 = 60$ min.
$x_3 = 75$ min.	$x_3 = 75$ min.
$x_4 = 93$ min.	$x_4 = 93$ min.
Sample size 4	Testing is stopped. The remaining 6 items have 93 min. of tested time but have not failed
$\hat{\lambda} = \frac{4}{258} = .0155$	$\hat{\lambda} = \frac{4}{816} = .005$
$\hat{R}(t) = e^{-.0155t}$	$\hat{R}(t) = e^{-.005t}$
$\hat{R}(10) = .86$	$\hat{R}(10) = .95$

Example 1.3 Type I censoring (fixed test time)

Under this test plan we put n items on test simultaneously and test until time T_o. All failed items are instantly replaced. All testing stops at time T_o.

Note 1: The total accumulated test time on all hardware tested, including replacement, is nT_o.

Note 2: The number of failures, R, is random.

Note 3: At all times in $(0,T_o)$ there are n items on test. The time between failures, Y, has an exponential distribution with parameter $n\lambda$, that is, $f_Y(y) = n\lambda\exp(-n\lambda y) \quad y > 0$.

Note 4: In view of Note 3, R, the number of failures, is Poisson with mean $n\lambda T_o$, that is, R is $P(n\lambda T_o)$.

Note 5: The total number of items put on test is n + R. The maximum likelihood estimate for λ is

$$\hat{\lambda} = \frac{r}{nT_o}$$

and for R(t) is $e^{-\hat{\lambda}t}$

Example 1.4 Random number of failures and random test time

In this case it is still possible to obtain a point estimate for λ and hence for $e^{-\lambda t}$, but to obtain confidence bounds for $e^{-\lambda t}$ is very difficult.

The maximum likelihood estimate for λ is

$$\hat{\lambda} = \frac{\text{number of failures}}{\text{total test time}} .$$

2. Point Estimates for the Gamma Distribution

See Methods for Statistical Analysis of Reliability and Life Data by Mann, Schafer, and Singpurwalla.

3. Point Estimates for the Weibull Distribution for Censored and Uncensored Data

See Methods for Statistical Analysis of Reliability and Life Data by Mann, Schafer, and Singpurwalla.

4. Point Estimates for the Mixed Exponential Distributions

See "Problems of Estimation for a Decreasing Failure Rate Distribution Applied to Reliability," by Myhre and Saunders (to appear in Technometrics).

5. Point Estimates for the Bernoulli distribution

If $x_1,\ldots,x_n$ is an observed sample from a Bernoulli distribution (go/no-go), then the maximum likelihood estimate for reliability p, is given by

$$\hat{p} = \frac{x_1 + \ldots + x_n}{n} = \frac{\text{number of successes}}{\text{number of items tested}}$$

For example if n = 10 and $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$, $x_5 = 1$, $x_6 = 1$, $x_7 = 1$, $x_8 = 0$, $x_9 = 1$, $x_{10} = 1$ then

$$\hat{p} = \frac{8}{10} .$$

V. CONFIDENCE BOUNDS FOR COMPONENT RELIABILITY

1. Review of Confidence Bounds

Let $X_1,\ldots\ldots,X_n$ be a random sample from a distribution with density function $f(x;\theta)$, where θ is an unknown parameter.

For example $X_1,\ldots,X_n$ could be a random sample from a normal distribution with known standard deviation σ_o and unknown mean θ.

$$f(x;\theta) = \frac{1}{\sigma_o\sqrt{2\pi}} \exp - \frac{(x-\theta)^2}{2\sigma_o^2} \qquad -\infty < x < \infty$$

or $X_1,\ldots,X_n$ could be a random sample from an exponential distribution with failure rate θ.

$$f(x;\theta) = \theta e^{-\theta x} \qquad x > 0.$$

Two sided confidence bounds for θ are given by the random interval determined by $B_L(X_1,\ldots,X_n)$ and $B_U(X_1,\ldots,X_n)$ where

$$P[B_L(X_1,\ldots,X_n) \le \theta \le B_U(X_1,\ldots,X_n)] = \beta .$$

β is called the confidence coefficient.

Example 1.1

Let $X_1,\ldots,X_n$ be a random sample from a normal distribution with mean θ and known standard deviation σ.

It can be proved that $Z = \dfrac{\bar{X}-\theta}{\sigma_o/\sqrt{n}}$ has a normal distribution with mean 0 and standard deviation 1, where $\bar{X} = \sum_{i=1}^{n} \frac{X_i}{n}$

The confidence interval is obtained from

$$P(-z \le \frac{\bar{X}-\theta}{\sigma_o/\sqrt{n}} \le z) = \beta$$

This is equivalent to

$$P\left(\bar{X} - \frac{z\sigma_o}{\sqrt{n}} \le \theta \le \bar{X} + \frac{z\sigma_o}{\sqrt{n}}\right) = \beta$$

Example 1.2 Larson, "Introduction to Probability Theory and Statistical Inference."

Suppose that the number of ounces which a machine puts into a bottle can be represented by a normal random variable with unknown mean θ and known standard deviation of 1/2 ounce. Select at random 25 bottles which have been filled by this machine. Let $X_1, X_2, \ldots, X_{25}$, respectively, denote the number of ounces they contain. Then $\overline{X}$ is a normal random variable with mean θ and standard deviation $1/2/\sqrt{25} = 1/10$. If we want 95% confidence bounds on θ then from the normal table

$$P(-1.96 \le \frac{\overline{X}-\theta}{1/10} \le 1.96) = .95$$

which is equivalent to

$$P(\overline{X} - 1.96 \frac{1}{10} \le \theta \le \overline{X} + 1.96 \frac{1}{10}) = .95$$

If the observed values of the random sample are:

$x_1 = 12.0$	$x_6 = 12.3$	$x_{11} = 12.4$	$x_{16} = 11.6$	$x_{21} = 11.9$
$x_2 = 12.8$	$x_7 = 11.5$	$x_{12} = 11.6$	$x_{17} = 11.9$	$x_{22} = 12.1$
$x_3 = 11.9$	$x_8 = 11.4$	$x_{13} = 12.1$	$x_{18} = 11.5$	$x_{23} = 12.3$
$x_4 = 11.8$	$x_9 = 12.2$	$x_{14} = 12.4$	$x_{19} = 12.0$	$x_{24} = 11.8$
$x_5 = 12.1$	$x_{10} = 13.0$	$x_{15} = 11.8$	$x_{20} = 12.2$	$x_{25} = 11.7$

Then

$$\overline{X} = 12.0$$

$$B_L = 12.0 - 1.96(1/10) = 12.0 - .2 = 11.8$$

$$B_U = 12.0 + 1.96(1/10) = 12.0 + .2 = 12.2$$

2. Confidence Bounds for the Exponential Distribution

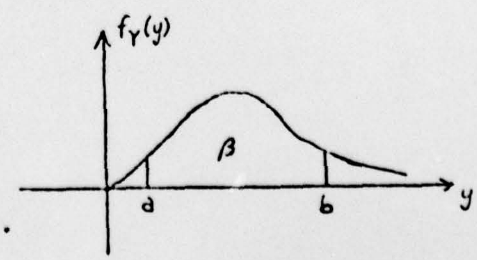
Example 2.1

Let $X_1, \ldots, X_n$ be a random sample from a distribution with probability density function

$$f(x;\theta) = \lambda e^{-\lambda x} \qquad x > 0$$

It can be shown that $Y = 2\lambda \sum_{i=1}^{n} X_i$ has a χ^2_{2n} distribution (a chi-square distribution with 2n degrees of freedom).

$$P(a < 2\lambda \sum_{i=1}^{n} X_i < b) = \beta \, .$$



This is equivalent to

$$P\left[\frac{a}{2\Sigma X_i} < \lambda < \frac{b}{2\Sigma X_i}\right] = \beta \ .$$

If the $X_1,\ldots,X_n$ denote life times for some particular piece of equipment then the reliability at time t is given by

$$R(t) = e^{-\lambda t} \ .$$

A confidence interval for R(t) is obtained by

$$P\left[e^{-bt/2\Sigma X_i} < e^{-\lambda t} < e^{-at/2\Sigma X_i}\right] = \beta \ .$$

Using the data from Example 1.1 of Section IV with $\beta = .90$ and $n = 4$, we have $a = 2.73$ and $b = 15.5$, a 90% confidence interval for $R(t) = e^{-\lambda t}$ is

$$(\exp -\frac{15.5}{2(258)}\, t,\ \exp -\frac{2.73}{2(258)}\, t)$$

$$= (e^{-.030t},\ e^{-.005t}) \ .$$

If t = 10 minutes then the 90% confidence interval for R(t) is (.74, .95).

Example 2.2 Type II censoring(fixed number of failures)

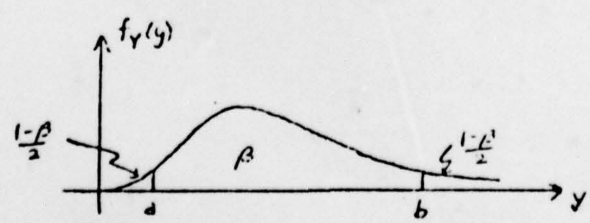
Let $X_{(1)} < X_{(2)} < \ldots . < X_{(n)}$ be an ordered sample from an exponential distribution with failure rate λ.

Assume Type II censoring at the r^{th} failure.

Let the total tested life, T, be given by

$$T = \sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)} \ .$$

Epstein and Sobel (1953) proved that $Y = 2\lambda T$ has a χ^2 distribution with 2r degrees of freedom.



$100\beta\%$ confidence bounds for λ are obtained from using the fact that

$$P(a < 2\lambda T < b) = \beta$$

$$P(a/2T < \lambda < b/2T) = \beta$$

Two sided confidence bounds for the reliability at time t are obtained from

$$P(e^{-bt/2T} < e^{-\lambda t} < e^{-at/2T}) = \beta \ .$$

A lower $\beta + (\frac{1-\beta}{2})$ bound is obtained from

$$P(e^{-(b/2T)t} < e^{-\lambda t}) = \beta + (\frac{1-\beta}{2}) \ .$$

Using the data from Example 1.2 of Section IV, the degrees of freedom for the χ^2 distribution are $2r = 8$. If $\beta = .90$ then $a = 2.73$ and $b = 15.5$.

Two sided <u>90%</u> confidence bounds for $e^{-\lambda t}$ are

$$e^{-.0095t} = e^{-\frac{15.5}{1632}} < e^{-\lambda t} < e^{-\frac{2.73}{1632}} = e^{-.0017t}$$

If t = 10 minutes then the bounds become (.91, .98) .

The one sided <u>95%</u> = $(90 + \frac{(100-90)}{2}$ lower bound is

$$e^{-.0095t} < e^{-\lambda t} \ .$$

Example 2.3 Type I censoring (fixed test time)

Under this test plan we put n items on test simultaneously and test until time T_o. All failed items are instantly replaced. All testing stops at time T_o. See Example 1.3 of Section IV.

It can be shown that if b is the 100β percentile point from a χ^2 distribution with 2(R+1) degrees of freedom then an upper 100β% confidence bound for λ is obtained by

$$P(2n\lambda T_o < b) = \beta$$

$$P(\lambda < \frac{b}{2nT_o}) = \beta$$

A lower 100β% confidence bound for $e^{-\lambda t}$ is obtained by

$$P(e^{-\frac{bt}{2nT_o}} < e^{-\lambda t}) = \beta$$

so that the lower bound on R(t) is $e^{-\frac{bt}{2nT_o}}$.

If $n = 8$, $T_o = 102$ and R is observed at $r = 3$ then

for $\beta = .95$, $b = 15.5$ (recall that the degrees of freedom on the χ^2 is $2(r+1)$ where r is observed number of failures in the fixed time nT_o). Thus,

$$\frac{bt}{2nT_o} = \frac{15.5}{1632}\,t \text{ and}$$

$$e^{-.0095t} < e^{-\lambda t} = R(t)$$

with probability .95.

3. <u>Confidence Bounds for Gamma Distribution</u>

   See <u>Methods for Statistical Analysis of Reliability and Life Data</u> by Mann, Schafer, and Singpurwalla.

4. <u>Confidence Bounds for the Weibull Distribution for Censored and Uncensored Data</u>

   See "An Exact Asymptotically Efficient Confidence Bound for Reliability in the Case of the Weibull Distribution," by Johns and Lieberman, <u>Technometrics</u>, Vol. 8, No. 1, Feb. 1966; also, <u>Methods for Statistical Analysis</u> . . . , by Mann, Schafer, and Singpurwalla.

5. <u>Confidence Bounds for Bernoulli Distribution</u>

   Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli distribution.

Example 5.1

Assume that $\sum_{i=1}^{n} x_i = k$, that is, k successes out of n trials. A 95% lower confidence bound, p_L, for p is the value of p for which

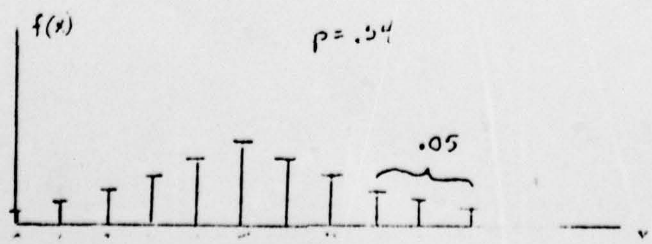
$$\sum_{y=k}^{n} \binom{n}{y} p^y (1-p)^{n-y} = .05$$

If $k = 0$ then $p_L = 0$.

To illustrate: if $n = 10$, $k = 8$ and confidence coefficient = .95 then

$$\sum_{y=8}^{10} \binom{10}{y} (.54)^y (.46)^{10-y} = .05$$

Thus if $p \le .54$ the probability of observing 8 or more successes in 10 trials is $\le .05$.



If $n = 5$, $k = 4$, confidence coefficient = .95

$$\sum_{y=4}^{5} \binom{5}{y} (.42)^y (.58)^{5-y} = .05$$

Example 5.2 Normal approximation

Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli distribution with probability density function

$$f(x) = p^x (1-p)^{1-x} \qquad x = 0.1$$

$\sum_{i=1}^{n} X_i$ has a Binomial distribution with parameters n and p.

By the Central Limit Theorem

$$\frac{\Sigma X_i - np}{\sqrt{np(1-p)}} \text{ is approximately } N(0,1),$$

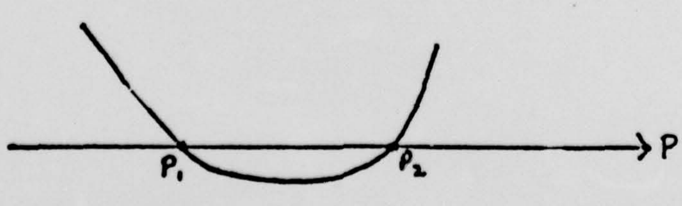
$$P\left[\frac{\Sigma X_i - np}{\sqrt{np(1-p)}} < z\right] \sim \int_{-z}^{z} (1/\sqrt{2\pi}) e^{-y^2/2}\, dy = \beta$$

$$\left|\frac{\Sigma X_i - np}{\sqrt{np(1-p)}}\right| < z \Leftrightarrow (\Sigma X_i - np)^2 < z^2 np(1-p)$$

$$\Leftrightarrow (\Sigma X_i)^2 - 2np(\Sigma X_i) + n^2p^2 < z^2np - z^2np^2$$

$$\Leftrightarrow (nz^2 + n^2)p^2 - (2n\Sigma X_i + nz^2)p + (\Sigma X_i)^2 < 0$$

This quadratic in p has 2 real roots



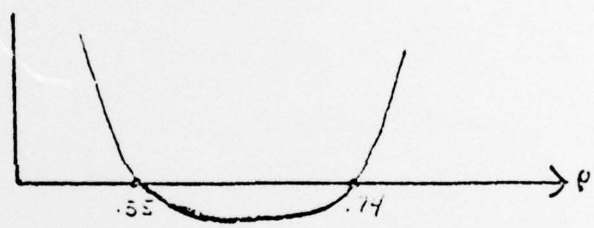
Solve for p_1, p_2. The confidence interval becomes (p_1, p_2). For example, if $n = 100$, $\Sigma X_i = 65$ and $\beta = .95$ then

$$P\left[-1.96 < \frac{\Sigma X_i - np}{\sqrt{np(1-p)}} < 1.96\right] \sim .95$$

$$\left|\frac{65-100p}{10\sqrt{p(1-p)}}\right| < 2 \quad \leftrightarrow \quad 10{,}400p^2 - 13{,}400p + 4225 < 0.$$

Roots are $p_1 = .55$

$p_2 = .74$



$(.55, .74)$ is an approximate 95% confidence interval for p.
.55 is a lower 97.5% confidence interval for p.

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ATTACHMENT F

PRELIMINARY DRAFT

APPROXIMATE CONFIDENCE BOUNDS FOR RELIABILITY: MIXED EXPONENTIAL DISTRIBUTION

by

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PRELIMINARY DRAFT

APPROXIMATE CONFIDENCE BOUNDS FOR RELIABILITY: MIXED EXPONENTIAL DISTRIBUTION

I. Introduction

The mixed exponential model for the probability that a component (or system) is successfully operating at time t, $R(t) = (1+\beta t)^{-\alpha}$, was developed for use where the exponential model had been assumed but the data showed a decreasing failure rate [2]. The maximum likelihood estimates of the parameters, either singularly or jointly, have been shown in [1] to be fairly accurate and in general preferable to using the best linear unbiased estimate of β. The estimates obtained by the method of moments are quite inaccurate.

Since the joint maximum likelihood estimate of α and β, $(\hat{\alpha},\hat{\beta})$, is not available in closed form, it is difficult to obtain the joint distribution of $(\hat{\alpha},\hat{\beta})$ and hence difficult to derive an exact lower confidence bound on reliability. It has been shown, however, that for fixed α and type II censoring of the data the distribution of $\hat{\beta}/\beta$ is independent of β. In addition, it has been shown [3] that for fixed α and general censoring (type II, type I or general time censoring) the distribution of $\hat{\beta}/\beta$ is asymptotically normal and independent of β as the sample size and number of failures become large. In these cases confidence bounds for β and hence for reliability can be obtained using either the simulated distribution of $\hat{\beta}/\beta$ or when appropriate the asymptotic normal distribution of $\hat{\beta}/\beta$.

If α is unknown, section IV gives a method for obtaining approximate confidence bounds for reliability. Simulations demonstrate the accuracy of these bounds as a function of sample size, number of failures and confidence level.

II. The Distribution of $\hat{\beta}$ when α is Known

For known scale parameter α it is shown in [2] that the distribution of $\hat{\beta}/\beta$ is independent of β, where $\hat{\beta}$ is the maximum likelihood estimate of β from a sample of size n with k failures. While this distribution is independent of β, it does depend on n, k, and α, thus it is necessary to know the distribution for each triple (n,k,α). If the distribution is known then upper (lower) bounds can be found for β since if $P[\hat{\beta}/\beta > b] = \alpha$ then $P[\hat{\beta}/\ \ > \beta] = \alpha$. From this expression lower confidence bounds on the reliability $R(t) = (1+\beta t)^{-\alpha}$ can be derived.

For each triple (n,k,α) with $\beta = 1$, 2500 $\hat{\beta}$'s were simulated and the resulting percentiles of the distribution obtained. These simulations were done for $\alpha = .05, .1, .2, \ldots, .9, 1.0, 1.2, \ldots 3.0$, $n = 25, 50, 100, 200$ and a range of k's appropriate for each n (i.e., for $n = 100$, $k = 5, 10, 20, \ldots 100$).

The graphs of these simulations appear to be approximately normal for all but small k and thus illustrate the result proven in [3] that the distribution of $\hat{\beta}/\beta$, for fixed α, is asymptotically $N(1,\sigma^2)$, where

$$\sigma^2 = \left[\sum_{i=1}^{k} \frac{\Gamma(n+1)\Gamma(n-i+1+2/\alpha)}{\Gamma(n-i+1)\Gamma(n+1+2/\alpha)}\right]^{-1} .$$

To determine the values of (n,k,α) for which the above approximation holds, one hundred simulation sets are run for each (n,k,α), where in each simulation set one hundred $\hat{\beta}$'s are estimated. Each simulation set sample of $\hat{\beta}$'s is tested by the Kolmogorov-Smirnov goodness of fit test for a $N(1,\sigma^2)$ distribution.

Compilation of computer results
to be inserted here.

In checking the distribution $\hat{\beta}/\beta$ the mean has been computed for each sample and is *always* slightly longer than one, say 1.023. Thus, even though the distribution appears to be normal it may be necessary to use a more accurate mean to get the proper fit.

III. Percentiles of the $\hat{\beta}/\beta$ Distribution

Since the percentiles of the $\hat{\beta}/\beta$ distribution depend on (n,k,α), the variation in the percentiles were studied as functions of α and k. For a fixed n and k it appeared that the 90th percentile point, b, was log log related to α. Linear regressions were run to fit $e^b=m(\ln\alpha) + c$ and these were found to be significant. Further study indicated that these 90th percentile points were linearly related to $1/\alpha$. When regressions were run for fixed n and k to fit $b=m(1/\alpha) + c$, where α varies through .05, .1, .2, . . . , .9, 1.0, 1.2, . . . 3.0, the results ere excellent with $R^2 > 99$. The bounds were also found to be linear functions of 1/k when n and α were fixed, and when regressions were run to fit $b=m(1/\alpha) + n(1/k) + c$ the results were good with multiple $R^2 > 90$. Similar results were found to hold for the 80th percentile points.

IV. Confidence Bounds on Reliability

With our knowledge of the distribution of $\hat{\beta}/\beta$ it is easy to construct a lower bound on reliability if the shape parameter, α, is known. If $P[\hat{\beta}/\beta > b] = \gamma$ then for all $t > 0$ and $\alpha > 0$ $P[(1+\hat{\beta}/bt)^{-\alpha} < (1+\beta t)^{-\alpha}] = \gamma$. Thus, if α is known a priori and if b can be determined for the particular n, k, and α, then $\underline{R}(t) = (1+\hat{\beta}/bt)^{-\alpha}$ is a lower confidence bound on the true reliability at a γ level of confidence.

Unfortunately in most applications α is unknown and α and β must be jointly estimated. In this case it is proposed that $\underline{R}(t) = (1+At)^{-\alpha}$ be used as the γ-confidence level lower bound on reliability where $A=\hat{\beta}/b_{(\hat{\alpha})}$ and $b(\hat{\alpha})$ is the γ-level bound on $\hat{\beta}/\beta$ assuming the true $\alpha = \hat{\alpha}$. If in fact $\hat{\beta}/b > \beta$ and $\hat{\alpha} > \alpha$ then $\underline{R}(t)$ will be a lower bound on $R(t)$ for all t, but in other cases $\underline{R}(t)$ may be lower than $R(t)$ for some t but not others. Thus the validity of $\underline{R}(t)$ must be shown on an appropriate time interval through computer simulation.

As $\hat{\alpha}$ is seldom one of the α for which the distribution of $\hat{\beta}/\beta$ is known, a major problem has been to calculate $b(\hat{\alpha})$. This has been solved by either using the linear model $b=m(1/\alpha) + c$, where the model is known for particular n and k, or by assuming that the distribution $\hat{\beta}/\beta$ is normal with $\mu = 1$ (or $1+\epsilon$) and

$$\sigma^2 = \left[\sum_{i=1}^{k} \frac{\Gamma(n+1)\Gamma(n+1-i+2/\alpha)}{\Gamma(n+1-i)\Gamma(n+1+2/\alpha)}\right]^{-1} .$$

In either case to test the validity of $\underline{R}(t)$ for a fixed α, β, n, k two hundred pairs of estimates $(\hat{\alpha},\hat{\beta})$ are generated and for each

pair $\underline{R}(t) = (1+\hat{\beta}/b_{(\alpha)}t)^{-\hat{\alpha}}$ is computed at each time t_p, p = 80, 81, . . . , 99 where $R(t_p) = p$. The $\underline{R}(t_p)$ are compared with the true reliability at each of the 20 times and the results averaged over all 200 pairs to determine the percentage of time that $\underline{R}$ is actually lower than the true reliability. As the percentage that results may be high because the bounds are too conservative, or be low because they are too optimistic, each $\underline{R}(t)$ is raised by .01 for all the t_p and lowered by .01 for all t_p and the corresponding percentages computed. For example, we simulated for n = 100, k = 30, α = .1, β = .3 two hundred $(\hat{\alpha},\hat{\beta})$ and generated 90% bounds $\underline{R}(t_p)$ for p = .80, .81, . . .,.99 using the linear model $b=m(1/\alpha) + c$. For this case,

$\underline{R}(t_p)$ is less than the true reliability 92% of time

$\underline{R}(t_p)-.01$ is less than the true reliability 96% of time

$\underline{R}(t_p)+.01$ is less than the true reliability 76% of time.

This shows a fairly tight lower bound. When using $b=\mu+\alpha_{10}\sigma$ on the assumption that the distribution is normal, the percentages are 95%, 98%, 88%, indicating that $\underline{R}(t)$ is quite conservative and the normal approximation does not fit well.

The following table indicates the accuracy of the approximate bounds and the value of n, k, and α for which the normal approximation bounds are accurate.

Compilation of computer Results
to be inserted here.

References

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